UNIT-1 PARTIAL DIFFERENTIAL EQUATIONS

Definition

A partial differential equation is an equation involving a function of two or more variables and some of its partial derivatives.

The order of a partial differential equation is the order of the highest partial derivative occurring in the equation.

Examples

1. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$ [u-dependent variable; x,y –independent variables]

2. $\left(\frac{\partial u}{\partial x}\right)^3 + \frac{\partial u}{\partial y} = 0$ [u- dependent Variable; x,y- independent variables]

3. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} = 0$ [u- dependent variable; x,y,t- independent variables]

In the above example, equation (1) is a second order equation; equation (2) is a first order equation (3) is also a first order equation.

Formation of partial Differential Equation

There are two methods two methods to form a partial differential equation.

(1) By elimination of arbitrary constants

(2) By elimination of arbitrary functions

1.1. By elimination of arbitrary constants- Method

Let us take the function f(x,y,z,a,b)=0….(1), where a & b are arbitrary constants.

Differentiate (1) partially with respect to the independent variables x,y we get,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \text{ …..(2)}$$

Similarly, $\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0$ …..(3)

Eliminating the two arbitrary constants a & b form (1), (2) & (3), we get a PDF of the first order of the form $\phi(x,y,z,p,q)=0$.

Note

If the number of arbitrary constants to be eliminated is equal to the number of independent variables, elimination of constants gives a first order partial differential equation. If the number of arbitrary constants to be eliminated is greater than the number of independent variables, then the elimination of constants gives second or higher orders partial differential equations.

Problems

1. Form the PDE by eliminating the arbitrary constants a & b form $z=(x^2+a)(y^2+b)$

Soln

Given $z=(x^2+a)(y^2+b)$…… (1)

Differentiate partially with respect to x & y we get,

$$\frac{\partial z}{\partial x}=2x(y^2+b)\Rightarrow p=2x(y^2+b)$$

$$\frac{\partial z}{\partial y}=2y(x^2+a)\Rightarrow q=2y(x^2+a)$$

$$\therefore x^2+a=\frac{q}{2y} \quad \& \quad y^2+b=\frac{p}{2x}$$

Substitute these in (1) we get

$$z=\frac{q}{2y}\frac{p}{2x}=\frac{pq}{4xy}$$

$$\Rightarrow 4xyz=pq$$

2. Find the PDF of all planes cutting equal intercepts from the x & y axis

Soln.

Let a, c be the intercepts on x& z axis respectively.

Hence, the equal of the plane is $\frac{x}{a}+\frac{y}{a}+\frac{z}{c}=1$ …..(1)

Differentiate (1) partially with respect to x and y, we get

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial x}=0 \quad \Rightarrow \quad \frac{1}{a}+p\frac{1}{c}=0 \text{ ……(2)}$$

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial y}=0 \quad \Rightarrow \quad \frac{1}{a}+q\frac{1}{c}=0 \text{ ……(3)}$$

(2)-(3) $\Rightarrow \frac{1}{c}(p-q)=0$

$\Rightarrow (p-q)=0$

1.2. By elimination of arbitrary functions- Method

The elimination of one arbitrary function from a given relation gives a PDE of first order while elimination of two arbitrary functions from a given relation gives second or higher order partial differential equations.

Problems

1.From the PDF by eliminating the arbitrary functions from (i). $z=f\left(x^2+y^2\right)$ &

(ii). $z=f(x+ct)+\phi(x-ct)$

Soln.

(i) Given $z=f\left(x^2+y^2\right)$

Differentiate partially with respect to x & y, we get

$$\frac{\partial z}{\partial x}=p=f'\left(x^2+y^2\right)2x$$

$$\frac{\partial z}{\partial y}=p=f'\left(x^2+y^2\right)2y$$

$\therefore \frac{p}{q}=\frac{2x}{2y} \Rightarrow py-qx=0$ is the required solution.

(ii). Given $z=f(x+ct)+\phi(x-ct)$

$$\frac{\partial z}{\partial x} = p = f^{'}(x+ct)+\phi^{'}(x-ct)$$

$$\frac{\partial^2 z}{\partial^2 x} = f^{''}(x+ct)+\phi^{''}(x-ct) \quad .....(1)$$

$$\frac{\partial z}{\partial t} = p = cf^{'}(x+ct)-c\phi^{'}(x-ct)$$

$$\frac{\partial^2 z}{\partial^2 t} = c^2 f^{''}(x+ct)+c^2\phi^{''}(x-ct) = c^2[f^{''}(x+ct)+\phi^{''}(x-ct)]$$

$\Rightarrow \frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ by equation (1).

$\therefore$ The required differential equation is $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$.

2. From the PDE by eliminating f from $z = x^2 + 2f\left(\frac{1}{y}+\log x\right)$

Soln

Given $z = x^2 + 2f\left(\frac{1}{y}+\log x\right)$ .....(1)

Differentiate partially with respect to x and y,

$$\frac{\partial z}{\partial x} = p = 2x + 2f^{'}\left(\frac{1}{y}+\log x\right)\frac{1}{x} \quad ....(2)$$

$$\frac{\partial z}{\partial y} = q = 2f^{'}\left(\frac{1}{y}+\log x\right)\left(\frac{-1}{y^2}\right) \quad ....(3)$$

From (2), $px = 2x^2 + 2f^{'}\left(\frac{1}{y}+\log x\right)$ .....(4)

From (3), $qy^2 = -2f^{'}\left(\frac{1}{y}+\log x\right)$ ......(5)

(4)+ (5) $\Rightarrow px + qy^2 = 2x^2$ is the required solution.

1.3. Formation of partial differential equations by elimination of arbitrary function 'ϕ' from $\phi(u,v)=0$ where 'u' & 'v' are functions of x,y & z.

Let $\phi(u,v)=0$ …(1) be a given function of u & v where u & v are functions of x,y & z.

Differentiate (1) partially with respect to x & y we get,

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x}=0 \quad …..(2)$$

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y}=0 \quad …..(3)$$

Here we want to eliminate ϕ. To eliminate ϕ it is enough if we eliminate $\frac{\partial \phi}{\partial u}$ & $\frac{\partial \phi}{\partial v}$ from (2) & (3).

Elimination of $\frac{\partial \phi}{\partial u}$ & $\frac{\partial \phi}{\partial v}$ from (2) & (3) gives

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}=0 \quad ……(4)$$

Where $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x}\&\frac{\partial v}{\partial y}$ are to be determined from u & v where u & v functions of x,y & z.

Problems

1.Form the PDE by eliminating the arbitrary functions from the relation $\phi(x^2+y^2+z^2,\ lx+my+nz)=0$

Soln

Given $\phi(x^2+y^2+z^2,\ lx+my+nz)=0$ ……(1)

Let $u = x^2 + y^2 + z^2$, $v = lx + my + nz$

$\therefore$ (1) is of the form $\phi(u,v) = 0$ **…..(2)**

The elimination of ϕ from equation (2), we get,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} 2x + 2zp & l + np \\ 2y + 2zq & m + nq \end{vmatrix}$$

$$\Rightarrow (2x + 2zp)(m + nq) - (2y + 2zq)(l + np) = 0$$

$$\Rightarrow 2xm + 2xnq + 2zpm + 2znpq - 2yl - 2ynp - 2zql - 2zqnp = 0$$
$$\Rightarrow (mz - ny)p + (nx - lz)q = yl - mx$$

Which is the required equation.

2.From the PDF by eliminating f from $f\left(x^2 + y^2 + z^2, x + y + z\right) = 0$

Soln

Given $f\left(x^2 + y^2 + z^2, x + y + z\right) = 0$ …….(1)

Let $u = x^2 + y^2 + z^2$, $v = x + y + z$

$\therefore$ (1) is of the form $f(u, v = 0)$…..(2)

The elimination of f from (2) gives

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2x + 2zp & 1 + p \\ 2y + 2zp & 1 + q \end{vmatrix} = 0$$
$$\Rightarrow (2x + 2zp)(1 + q) - (2y + 2zq)(1 + p) = 0$$
$$\Rightarrow (x + zp)(1 + q) - (y + zq)(1 + p) = 0$$
$$\Rightarrow x + xq + zp + zpq - y - yp - zq - zpq = 0$$
$$\Rightarrow (z - y)p + (x - z)q = y - x$$

This is the required equation.

1.4. Definition

The equation $P_p + Q_q = R$ is called Lagrange's linear equation. Here $P = \frac{\partial(u,v)}{\partial(y,z)} = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$; $Q = \frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$ & $R = \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$.

Solution of partial differential equation by direct Integration

A PDE can be solved by successive integration in all cases where the dependent variable occurs only in the partial derivatives.

Note

In general, it our integration is with respect to x or y we can replace the arbitrary constants buy arbitrary functions of y or x respectively.

Problems

1.Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x$

Soln

Given $\frac{\partial^2 z}{\partial x \partial y} = \sin x$

Integrating w.r.t x we get

$\frac{\partial z}{\partial y} = -\cos x + f(y)$, where f is arbitrary . Again integrating w.r.t y we get

$z = -y\cos x + F(y) + \phi(x)$ where F & ϕ are arbitrary.

2.solve $\frac{\partial^2 z}{\partial x^2} = xy$

Solno

Given $\frac{\partial^2 z}{\partial x^2} = xy$

Integrating with respect to x, $\frac{\partial z}{\partial x} = \frac{x^2 y}{2} + f(y)$

Integrating with respectto y, $z = \frac{x^3 y}{6} + xf(y) + \phi(y)$ where f & ϕ are arbitrary.

Solution of Partial Differential Equations

Definition

A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution.

In complete integral if we give particular values to the arbitrary constants we get particular integral.

Definition

Let $f(x, y, z, p, q) = 0$ be a PDE whose complete integral is $\phi(x, y, z, a, b) = 0....(1)$ differentiate partially with respect to a & b then equal to zero we get $\frac{\partial \phi}{\partial a} = 0...(2)$ $\frac{\partial \phi}{\partial b} = 0...(3)$

The elimination of a & b from the above three equation is called singular integral.

Definition

A solution of a PDE which contains the maximum possible number of arbitrary functions is called a general integral or general solution.

Procedure to find general integral of the PDE

Let the PDE be $f(x, y, z, p, q) = 0.....(1)$. Let the complete integral of (1) be $\phi(x, y, z, a, b) = 0.......(2)$ where a & b are arbitrary constants.

Suppose in (2) one of the constants is a function of the other say $b = f(a)$.

Then (2) becomes $\phi(x, y, z, a, f(a) = 0......(3)$. Differentiate (3) partially with respect to a we get

$\frac{\partial \phi}{\partial a} + \frac{\partial u}{\partial a} f'(a) = 0........(4)$. The elimination of 'a' between (3) & (4) if it exist is called the general integral of (1).

Methods to solve the first order PDE

Type:1 $f(p, q) = 0$

Type :2 $z = px + qy + f(p, q)$ **(clairauts form)**

Type:3 $f(z, p, q) = 0; f(x, p, q) = 0; f(y, p, q) = 0$

Type:4 $f(x,p)=\phi(y,q)$

Type:5 $f(x^m p, y^n q)=0$ **and** $f(z,x^m p, y^n q)=0$

Type : 6 $f(z^m p, z^m q)=0$ **and** $f_1(x,z^m p)=f_2(y,z^m q)$

Type 1: $f(p,q)=0$

The equation contains p & q only. Suppose that $z=ax+by+c$ is a solution of the equation $f(p,q)=0$.

Suppose that, $z=ax+by+c$ is a trial solution of $f(p,q)=0$. Then $p=\frac{\partial z}{\partial x}=a,\ q=\frac{\partial z}{\partial y}=b$ we get $f(a,b)=0$.

Hence, the complete solution of the given equation is $z=ax+by+c$ where $f(a,b)=0$. Solving for b from $f(a,b)=0$.

We get $b=\phi(a)$ say then $z=ax+\phi(a)y+c$ is the complete integral of the given equation.

Since, it contains two arbitrary constants, singular integral is got by eliminating a &c from $z=ax+\phi(a)y+c$.......(1)

Differentiating partially with respect to a, we get

$0=x+\phi'(a)y$

Differentiating partially with respect to c, we get

0=1 which is absurd

There is no singular integral for the given PDE

To find the general integral put $c=f(a), f$ being arbitrary.

Then $z=ax+y\phi(a)+f(a)$........(2)

Differentiate partially with respect to a, we get

$0=x+y\phi'(a)+f'(a)$.....(3)

Eliminating ‘a’ between (2) and (3) we get the general solution.

Problem based on Type 1

1.solve $\sqrt{p}+\sqrt{q}=1$

Soln

$\sqrt{p}+\sqrt{q}=1$

This is of the form $F(p,q)=0$

Hence complete integral is $z=ax+by+c$ where $\sqrt{a}+\sqrt{b}=1$

That is $\sqrt{b}=1-\sqrt{a} \Rightarrow b=\left(1-\sqrt{a}\right)^2$

The complete solution is $z=ax+\left(1-\sqrt{a}\right)^2 y+c......(1)$

Differentiate partially with respect to c, we find that there is no singular solution

Taking $c=f(a)$ where f is arbitrary.

$z=ax+\left(1-\sqrt{a}\right)^2 y+f(a).....(2)$

Differentiate (2) partially with respect to ‘a’ we get

$$0=x+2y(1-\sqrt{a}).(\frac{-1}{2}a^{\frac{-1}{2}})+f^{'}(a)$$

$$x-2y(1-\sqrt{a}).(\frac{1}{2\sqrt{a}})+f^{'}(a)=0$$

$$x-y\frac{(1-\sqrt{a})}{\sqrt{a}}.+f^{'}(a)=0.......(3)$$

Eliminating ‘a’ between (2) & (3) we get the general integral.

Type II (Clairauts form $z=px+qy+f(p,q)$

Suppose that the given equation is of the form $z=px+qy+f(p,q)$ …..(1)

The complete solution is $z=ax+by+f(a,b)........(2)$ where ‘a’ ‘b’ are arbitrary constants.

Differentiate (20 partially with respect to ‘a’ & ‘b’ we get

$$x+\frac{\partial f}{\partial a}=0......(3) \text{ and } y+\frac{\partial f}{\partial b}=0......(4)$$

By eliminating a & b from (2), (3) & (4) we get the singular integral of (1)

Taking $b = \phi(a)$, (2) becomes

$z = ax + \phi(a)y + f[a, \phi(a)]$........(5)

Differentiate partially with respect to 'a' we get

$0 = x + \phi'(a)y + f'[a, \phi(a)]$..........(6)

Eliminating a between (5) & (6) we get the general integral of (1).

Problem based on Type II

Solve $z = px + qy + p^2q^2$

Soln

Given $z = px + qy + p^2q^2$

The complete solution is $z = ax + by + a^2b^2$........(1)

Differentiate partially with respect to 'a' & 'b'

$0 = x + 2ab^2$ **&** $0 = y + 2a^2b$

Therefore $x = -2ab^2$.......(2) & $y = -2a^2b$.......(3)

$\Rightarrow \frac{x}{b} = -2ab$ & $\frac{y}{b} = -2ab$

$\therefore \frac{x}{b} = \frac{y}{a} = -2ab = \frac{1}{k}$ (say)

$\Rightarrow a = ky$ & $b = kx$ put in (2)

$x = -2kyk^2x^2 \Rightarrow x = -2k^3x^2y$

$k^3 = \frac{-1}{2xy}$ ……..(4)

put a & b in (1)

$z = kxy + kxy + k^2y^2k^2x^2$

$= 2kxy + k^4 x^2 y^2$

$= 2kxy + k \ x^2 y^2 \left(\frac{-1}{2xy} \right)$ by (4)

$= 2kxy - \frac{kxy}{2} = \frac{3}{2} kxy$

$\therefore z^3 = \frac{27}{8} k^3 x^3 y^3$

$= \frac{27}{8} x^3 y^3 \left(\frac{-1}{2xy} \right)$

$z^3 = \frac{-27}{16} x^2 y^2$

$16z^3 + 27x^2 y^2 = 0$ is the singular solution put $b = f(a)$ in (1).

$z = ax + f(a)y + a^2 [f(a)]^2$.............(5)

Differentiate partially with respect to 'a'

$0 = x + f'(a)y + 2a^2 f(a) f'(a) + 2a[f(a)]^2$

$\Rightarrow x + f'(a)y + 2a^2 f(a) f'(a) + 2a(f(a))^2 = 0$.......(6)

Eliminating 'a' between (5) and (6) we get the general solution.

Type 3

(a) Equation of the type $F(z, p, q) = 0$**. That is equation not containing x and y.**

Let z be a function of $u = x + ay$ **where 'a' is an arbitrary constant.**

Now $z = f(u) = f(x + ay)$

Then $p = \frac{\partial z}{\partial x} = \frac{dz}{du} . \frac{\partial u}{\partial x} = \frac{dz}{du} .1 = \frac{dz}{du}$

$q = \frac{\partial z}{\partial y} = \frac{dz}{du}.\frac{\partial u}{\partial y} = \frac{dz}{du}.a = a\frac{dz}{du}$ substitute these values of p & q in F(z,p,q)=0. We get $F(z, \frac{dz}{du}, a\frac{dz}{du}) = 0$ which is an ordinary differential equation of the first order. Solving for $\frac{dz}{du}$, we obtain $\frac{dz}{du} = \phi(z,a)$ (say)

$$\frac{dz}{\phi(z,a)} = du$$

Integrating we get , $\int \frac{dz}{\phi(z,a)} = u + c$

$f(z,a) = u + c \Rightarrow f(z,a) = x + ay + c$. This is the complete integral.

(b) Equation of the type $F(x, p, q) = 0$**. Since z is a function of x & y.**

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$

Assume that q=a

Then the equation becomes $F(x, p, a) = 0$. Solving for p, we obtain $p = \phi(x,a)$

$dz = \phi(x,a)dx + ady$.

Hence $z = \int \phi(x,a)dx + ay + c \Rightarrow z = f(x,a) + ay + c$ .This is the complete integral.

(c) Equation of the type $F(y, p, q) = 0$**. Assume p=a, solving for p we obtain** $q = f(y,a)$

$\therefore dz = adx + f(y,a)dy$

Hence $z = ax + \int f(y,a)dy + c \Rightarrow z = ax + g(y,a) + c$

This is the complete integral.

Problems based on Type 3

Solve $9(p^2 z + q^2) = 4$

Soln

This is of the form $F(z, p, q) = 0$

Assume $\left(\dfrac{dz}{du}\right) = \quad u = x + ay$

Then $p = \dfrac{dz}{du}$ and $q = a\dfrac{dz}{du}$ substitute this values in the given equation

$$9\left[\left(\frac{dz}{du}\right)^2 z + a^2\left(\frac{dz}{du}\right)^2\right] = 4$$

$$\left(\frac{dz}{du}\right)^2 \left[9(z+a^2)\right] = 4 \Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{4}{9(z+a^2)}$$

$$\left(\frac{dz}{du}\right) = \frac{2}{3}\frac{1}{\sqrt{z+a^2}}$$

$$3\sqrt{z+a^2}\,dz = 2du$$

Integrating, $3\int \sqrt{z+a^2}\,dz = 2u + 2c$

$$\Rightarrow 3\frac{\left(z+a^2\right)^{3/2}}{3/2} = 2u + 2c$$

$$\Rightarrow 2\left(z+a^2\right)^{3/2} = 2u + 2c..........(*)$$

$\Rightarrow \left(z+a^2\right)^3 = \left(u+c\right)^2 \Rightarrow \left(z+a^2\right)^3 = \left(x+ay+c\right)^2........(1)$. This is the complete soln

Differentiate (*) partially with respect to c, we get 0=1 which is absurd

Hence there is no singular integral.

To find general integral, put $c = \phi(a)$ in (1)

$$\left(z+a^2\right)^3 = \left(x+ay+\phi(a)\right)^2........(2)$$

Differentiate partially with respect to ‘a’

$$3\left(z+a^2\right)^2.2a = 2\left(x+ay+\phi(a)\right).\left(y+\phi^{'}(a)\right)$$

$6a\left(z+a^2\right)^2 = 2\left(x+ay+\phi(a)\right)\left(y+\phi^{'}(a)\right)$.......(3) eliminate a between (2) and (3) we get the general integral.

Type 4 separable equations

A first order partial differential equation is separable if it can be written us $f(x,p)=\phi(y+q)$. Put each of these equal to an arbitrary constant a (say).

Hence $f(x,p)=\phi(y+q)=a$

Solving for p and q, we get $p=f_1(x,a)\,\&\,q=\phi_1(y+a)$. Since $dz=pdx+qdy=f_1(x,a)dx+\phi_1(y+a)dy$

$\therefore z=\int f_1(x,a)dx+\int \phi_1(y+a)dy+b$. This is the required complete integral.

Problem based on Type 4

1.Solve $p^2(1+x^2)=qx^2$

Soln

Given $p^2(1+x^2)=qx^2$.

The equation is separable

$\frac{p^2(1+x^2)}{x^2}=\frac{q}{y}=a$ where a is an arbitrary constant.

Thus $\frac{p^2(1+x^2)}{x^2}=a \Rightarrow p^2=\frac{a^2}{1+x^2} \Rightarrow p=\frac{x\sqrt{a}}{\sqrt{1+x^2}}$

Again $dz=pdx+qdy$

$$dz=\frac{x\sqrt{a}}{\sqrt{1+x^2}}dx+aydy$$

Integrating, $z=\sqrt{a}\int\frac{x}{\sqrt{1+x^2}}dx+\frac{ay^2}{2}+b$

$$\Rightarrow \sqrt{a}\int\frac{dt/2}{\sqrt{t}}+\frac{ay^2}{2}+b$$

$$\frac{2\sqrt{a}\sqrt{t}}{2}+\frac{ay^2}{2}+b=\sqrt{a\left(1+x^2\right)}+\frac{ay^2}{2}+b$$

is the complement integral.

Differentiate partially with respect to b, we get 0=1 which is absurd

Hence there is no singular integral

To find the general integral put b=f(a) in (1)

$$z=\sqrt{a\left(1+x^2\right)}+\frac{ay^2}{2}+f(a).......(2)$$

Differentiate with respect to a we get

$$0=\sqrt{\left(1+x^2\right)}\frac{1}{2\sqrt{a}}+\frac{y^2}{2}+f'(a)$$

$$\sqrt{\left(1+x^2\right)}\frac{1}{2\sqrt{a}}+\frac{y^2}{2}+f'(a)=0.........(3)$$

Eliminating a between (2) and (3) we get the general integral.

Equations reducible to standard forms

Case 1 $F\left(x^m p, y^n q\right)=0$ **where m & n are constants**

Put $x^{1-m}=X$ and $y^{1-n}=Y$ where $m\neq 1 \,\&\, n\neq 1$ we get $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial x}=\frac{\partial z}{\partial X}.\frac{dX}{dx}=(1-m)x^{-m}\frac{\partial z}{\partial X}=(1-m)x^{-m}P$ &

$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}=\frac{\partial z}{\partial Y}.\frac{dY}{dy}=(1-n)x^{-n}\frac{\partial z}{\partial Y}=(1-n)x^{-n}Q$ where $P=\frac{\partial z}{\partial X}$ and $Q=\frac{\partial z}{\partial Y}$

Hence the given equation reduces to $F\left((1-m)P,(1-n)Q\right)=0$ which is of the form $F(P,Q)=0$

Case 2 $F\left(x^m p, y^n q, z\right)=0$ **where m & n are constants**

Put $x^{1-m}=X$ and $y^{1-n}=Y$ where $m\neq 1 \,\&\, n\neq 1$ we get $p=(1-m)x^{-m}p$ &

$q = (1-n)y^{-n}Q$ where $P = \frac{\partial z}{\partial X}$ and $Q = \frac{\partial z}{\partial Y}$

Hence the given equation reduces to $F((1-m)P,(1-n)Q) = 0$ which is of the form $F(z,P,Q) = 0$.

Case 3 $F(xp, yq) = 0$ **or** $F(xp, yq, z) = 0$

In the above cases if m=1, put $X = \log x$

If n=1, put $Y = \log y$

Hence we get $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}.\frac{dX}{dx} = \frac{\partial z}{\partial X}\frac{1}{x} = P\frac{1}{x} = Px$ &

$q = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y}.\frac{dXY}{dy} = \frac{\partial z}{\partial Y}\frac{1}{y} = Q\frac{1}{x}$

$Q = qy$

$F(xp, yq) = 0$ becomes $F(P,Q) = 0$ &

$F(xp, yq, z) = 0$ becomes $F(P,Q,z) = 0$

Problem

Solve $p^2 + x^2y^2q^2 = x^2z^2$

Soln

Given $p^2 + x^2y^2q^2 = x^2z^2$

Divide by x^2**,** $(p\ x^{-1})^2 + y^2q^2 = z^2$.......(1)

This is of the form $F(px^m, qy^n, z) = 0$.

Here m=1 &n=1

Put $X = x^{1-m}$ and $Y = y^{1-n}$

$X = x^2$ & $Y = \log y$

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}.2x \Rightarrow x^{-1}p = 2P$

By using $x^m p = (1-m)P, x^{-1} p = 2P$

$$q == \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y}.\frac{1}{y} \Rightarrow qy = Q$$

$\therefore$ (1) becomes $(2P)^2 + Q^2 = Z^2$

$\Rightarrow 4P^2 + Q^2 = Z^{2.}$.......(2)

This is of the form $F(z, P, Q) = 0$

Let $z = f(u)$ where $u = X + aY$

$$\therefore P = \frac{dz}{du}, Q = a\frac{dz}{du}$$

Therefore (2) becomes , $4\left(\frac{dz}{du}\right)^2 + az\left(\frac{dz}{du}\right)^2 = z^2$

$$\left(\frac{dz}{du}\right)^2 \left[4 + a^2\right] = z^2$$

$$\Rightarrow \frac{dz}{du} = \frac{z}{\sqrt{4 + a^2}}$$

$$\Rightarrow \frac{dz\sqrt{4 + a^2}}{z} = du$$

Integrating we get $\sqrt{4 + a^2} \log z = u + b$

$$\sqrt{4 + a^2} \log z = X + aY + b$$

$\sqrt{4 + a^2} \log z = x^2 + a \log y + b$ is the complete soln.

Case 4 $F(z^k p, z^k q) = 0$ **where k is any constant**

If $k \neq -1$Put $Z = z^{k+1}$

Then $\frac{\partial z}{\partial x} = (k+1)z^p$ & $\frac{\partial z}{\partial y} = (k+1)z^k q$. Hence the given equation reduces to the form $P = \frac{\partial z}{\partial x}$ & $Q = \frac{\partial z}{\partial y}$ if $k = -1$ put $Z = \log z$

$\therefore \frac{\partial z}{\partial x} = \frac{1}{z} p$ & $\frac{\partial z}{\partial y} = \frac{1}{z} q$. Hence the given equation again reduces to the form $f(p,q) = 0$, where $p = \frac{\partial z}{\partial x}$ & $Q = \frac{\partial z}{\partial y}$

Case 5 $F(x^m z^k p, y^n z^k q) = 0$

Put $X = x^{1-m}, Y = y^{1-n}$ & $Z = z^{k+1}$ if $m \neq 1, n \neq 1$ & $k \neq 1$. Hence the given equation reduces to the form $F(P,Q) = 0$

Put $X = \log x, Y = \log y$ & $Z = \log z$ if $m = 1, n = 1$ & $k = 1$

Therefore the given equation again reduces to the form $F(P,Q) = 0$.

Problem

Solve $2x^4 p^2 - yzq - 3z^2 = 0$

Solution

$2x^4 p^2 - yzq - 3z^2 = 0$

This can be written as $2\left(\frac{x^2 p}{z}\right)^2 - \frac{yq}{z} - 3 = 0......(1)$

Here $m = 2, n = 1$ & $k = -1$

Put $X = x^{1-m}, Y = \log z$ & $Z = \log z \Rightarrow X = x^{-1}, Y = \log y$ & $Z = \log z$

$$P = \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z}.\frac{\partial z}{\partial x}.\frac{\partial x}{\partial X} = \frac{1}{z} p(-x^2) \Rightarrow \frac{px^2}{z} = -P$$

$$Q = \frac{\partial Z}{\partial Y} = \frac{\partial Z}{\partial z}.\frac{\partial z}{\partial y}.\frac{\partial y}{\partial Y} = \frac{1}{z} qy \Rightarrow \frac{yq}{z} = Q$$

Therefore (1) becomes $2(-P)^2 - Q - 3 = 0 \Rightarrow 2P^2 - Q - 3 = 0......(2)$

This is of the form $F(P,Q)=0$

The complete integral is $Z=aX+bY+c$ where $F(a,b)=0$

$2a^2-b-3=0 \Rightarrow b=2a^2-3$

$\therefore z=aX+(2a^2-3)Y+c$

$\Rightarrow \log z = ax^{-1}+(2a^2-3)\log y+c$

$\log z=\frac{a}{x}+(2a^2-3)\log y+c$ is the complete integral

Lagranges Linear Equation

A linear partial differential equation of the first order known as lagranges linear equation is of the form $P_p+Q_q=R$ where P,Q,& R are functions of x,y,z.

Methods to solve the equation $P_p+Q_q=R$

(i) Form the auxiliary simultaneous equation $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

(ii) Solve the auxiliary simultaneous equation giving two independent solution $u=a \,\&\, v=b$

(iii)Then write down the solution as $\phi(u,v)=0$ or $u=f(v)$ or $v=f(u)$ where the function is arbitrary.

Problems based on lagranges linear Equation

1.Find the general integral of $px+qy=z$

Soln

$px+qy=z$

Here $P=x, Q=y, R=z$

The subsidiary equation are $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$

Now, $\frac{dx}{x}=\frac{dy}{y}$

Integrating, $\log x=\log y=\log a$

$$\log x - \log y = \log a \Rightarrow \log\left(\frac{x}{y}\right) = \log a$$

$$\frac{x}{y} = a$$

Similarly from $\frac{dy}{y} = \frac{dz}{z}$ we get $\log y = \log z + \log b$

$$\log\left(y/z\right) = \log b$$

$\frac{y}{z} = b$. Hence the general integral is $\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

2.Find the general solution of $p \tan x + q \tan y = \tan z$

Solution

$p \tan x + q \tan y = \tan z$

here $P = \tan x, Q = \tan y$ & $R = \tan z$

The auxiliary equations are $\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$

$$\Rightarrow \cot xdx = \cot ydy = \cot zdz$$

From $\cot xdx = \cot ydy$, we get

$$\int \cot xdx = \int \cot ydy \Rightarrow \log \sin x = \log \sin y + \log a$$

$$\Rightarrow \frac{\sin x}{\sin y} = a$$

Taking the last two ratios, we get

$$\Rightarrow \int otydy = \int \cot zdz \Rightarrow \log \sin y = \log \sin z + \log b$$

$$\Rightarrow \frac{\sin y}{\sin z} = b$$

Hence the general soln is $\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

Linear partial differential equations of second and higher order with constant coefficients are constants.

where $a_0, a_1, a_2, a_3......a_n$ are constants.

$$\left[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + ... + a_n D'^n\right] z = F(x, y) \text{ where } D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$$

Methods to find complementary function

The auxiliary equation of $f(D, D')Z = F(x, y)$ is $f(m,1) = 0$. Since $D \to m, D' \to 1$.

The auxiliary equation is $a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + ...a_n = 0$

Case (i) If the roots are real (or imaginary) and different

$m_1 \neq m_2 \neq m_3 \neq .... \neq m_n$ the complementary function is

$$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \phi_3(y + m_3 x) + ... + .\phi_n(y + m_n x)$$

Case (ii) If the roots are equal

$m_1 = m_2 = m_3 = ... = m_n$ the complementary function is

$$z = \phi_1(y + mx) + x\phi_2(y + mx) + x^2\phi_3(y + mx) + ... + .x^n\phi_n(y + mx).$$

Methods of finding particular Integral

(i) **Problems based on R.H.S** $= e^{ax+by}$ **replace D by a and** D' **by b**

Solve $(D + D')^2 z = e^{x-y}$

Solution

Given $(D + D')^2 z = e^{x-y}$

The auxiliary equation is $(m+1)^2 = 0$ (replace D by m and D' by 1)

$m = -1, -1$. Here the roots are equal

The complementary function is $=\phi_1(y-x)+\phi_2(y-x)$

Particular integral is $=\dfrac{1}{\left(D+D^{'}\right)^2}e^{x-y}$........(1)

$$= =\frac{1}{\left(1+(-1)\right)^2}e^{x-y} \quad \text{replace D by 1 and D' by -1}$$

$$=\frac{1}{0}e^{x-y} \qquad \text{Ordinary rule fails}$$

$$=x\frac{1}{2\left(D+D^{'}\right)}e^{x-y}.........(2)$$

$$=\frac{x}{2}\frac{1}{\left(1+(-1)\right)}e^{x-y}$$

$$=\frac{x}{2}\frac{1}{o}e^{x-y} \quad \text{Again ordinary rule fails}$$

$$=\frac{x^2}{2}\frac{1}{-}e^{x-y}$$

Hence the general soln is z=C.F+P.I

$$=\phi_1(y-x)+x\phi_2(y-x)+\frac{x^2}{2}e^{x-y}$$

(ii) Problems based on R.H.S $=e^{ax+by}+x^r y^s$

Solve $\left(D^2+DD'-6D'^2\right)Z=x^2y+e^{3x+y}$

Solution

The auxiliary equation is $m^2+m-6=0$

$m=2,-3$

The complementary function is $=\phi_1(y-2x)+\phi_2(y-3x)$

$P.I=P.I_1+P.I_2$

$$P.I_1 = \frac{1}{D^2 + DD' - 6D'^2} x^2 y$$

$$= \frac{1}{D^2\left[1 + \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right)\right]} x^2 y$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right)\right] x^2 y$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right) + ...\right] x^2 y$$

$$= \frac{1}{D^2}\left[1 - \frac{D'}{D}\right] x^2 y$$ Omitting D'^2 and higher derivative

$$= \frac{1}{D^2}\left[x^2 y - \frac{1}{D}(x^2)\right]$$

$$= \frac{1}{D^2}(x^2 y) - \frac{1}{D^3}(x^2)$$

$$= y\left(\frac{x^4}{12}\right) - \frac{x^5}{60}$$

$$= \left(\frac{yx^4}{12}\right) - \frac{x^5}{60}$$

$$P.I_2 = \frac{1}{D^2 + DD' - 6D'^2} e^{3x+y}$$

$$= \frac{1}{9+3-6} e^{3x+y}$$

$$= \frac{1}{6} e^{3x+y}$$

Hence the general solution is

$$z = C.F + P.I_1 + P.I_2$$

$$z = \phi_1(y+2x) + \phi_2(y-3x) + \left(\frac{yx^4}{12}\right) - \frac{x^5}{60} + \frac{1}{6}e^{3x+y}$$

(iii)Problems based on R.H.S $= \sin(ax+by)$ **or** $= \cos(ax+by)$ Replace D^2 by $-a^2$, DD' by $-ab$, D'^2 by $-b^2$

Solve $\left(D^2 - 3DD' + 2D'^2\right)Z = \sin x \cos y$

Solution

The auxiliary equation is $m^2 - 3m + 2 = 0$

$m = 1,2$

$C.F = \phi_1(y+x) + \phi_2(y+2x)$

$$P.I = \frac{1}{D^2 - 3DD' + 2D'^2}\sin x \cos y$$

$$P.I = \frac{1}{D^2 - 3DD' + 2D'^2}\frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

$$P.I_1 = \frac{1}{D^2 - 3DD' + 2D'^2}\frac{1}{2}\sin(x+y)$$

$$P.I_1 = \frac{1}{2} I.Pof \frac{1}{D^2 - 3DD' + 2D'}e^{i(x+y)}$$

$= \frac{1}{2} I.Pof \frac{1}{(i)^2 - 3(i)(i) + 2(i)^2}e^{i(x+y)}$ Replace D by I and D’ b Replace D by I and D’ by i

$$= \frac{1}{2} I.Pof \frac{1}{-1+3-2}e^{i(x+y)}$$

$= \frac{1}{2} I.Pof \frac{1}{0}e^{i(x+y)}$ ordinary rule fails

$$= \frac{1}{2} xI.Pof \frac{1}{2D - 3D'}e^{i(x+y)}$$

$$= \frac{x}{2} I.Pof \frac{1}{2i - 3i}e^{i(x+y)}$$

$= \frac{x}{2} I.P of \frac{1}{-i} e^{i(x+y)}$

$= \frac{x}{2} I.P of \{i[\cos(x+y) + i\sin(x+y)]\}$

$= \frac{x}{2}\cos(x+y)$

$P.I_2 = \frac{1}{2}\left[\frac{1}{D^2 - 3DD' + 2D'^2}\sin(x-y)\right]$

$= \frac{1}{2}\left[\frac{1}{-1-3(1)+2(-1)}\sin(x-y)\right]$

$= \frac{1}{2}\left[\frac{1}{-4-2}\sin(x-y)\right]$

$= \frac{1}{-12}[\sin(x-y)]$

Hence the general soln is

$z = C.F + P.I_1 + P.I_2$

$z = \phi_1(y+x) + \phi_2(y+2x) + \frac{1}{2}[x\cos(x+y)] + \frac{1}{-12}[\sin(x-y)]$

(iv)Problems based on R.H.S $= \sin(ax+by)$ **+** e^{ax+by}

Solve $(D^2 - 7DD' - 6D'^2)Z = \sin(x+2y) + e^{3x+y}$

Solution

The auxiliary equation is $m^3 - 7m - 6 = 0$

$m = -1, -2, 3$

$C.F = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$

$P.I_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin(x+2y)$

$$P.I_1 = I.P of \frac{1}{D^3 - 7DD'^2 - 6D'^3} e^{i(x+2y)}$$

$$= \frac{1}{2} I.P of \frac{1}{(i)^2 - 3(i)(i) + 2(i)^2} e^{i(x+y)}$$ Replace D by i and D' by 2i

$$= I.P of \frac{1}{-i + 28i + 48i} e^{i(x+2y)}$$

$$= I.P of \frac{1}{75i} e^{i(x+2y)}$$

$$= I.P of \frac{1}{-75'} ie^{i(x+2y)}$$

$$= \frac{-1}{75} \cos(x+2y)$$

$$P.I_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} e^{3x+y}$$

$$= \frac{1}{0} e^{3x+y}$$ ordinary rule fails

$$= x \frac{1}{3D^2 - 7D'^2} e^{3x+y}$$

$$= x \frac{1}{27 - 7} e^{3x+y}$$

$$= x \frac{1}{20} e^{3x+y}$$

Hence the general solution is

$$z = C.F + P.I_1 + P.I_2$$

(v)Problems based on R.H.S $e^{ax+by}\phi(x, y)$ replace D by D+a, D' by D'+b

Solve $\left(D^3 + D^2D' - DD'^2 - D'^3\right)Z = e^x \cos 2y$

Solution

The auxiliary equation is $m^3+m^2-6m-1=0$

$(m+1)^2(m-1)=0m=1,-1,-1$

$C.F=\phi_1(y+x)+\phi_2(y-x)+\phi_3(y-x)$

$$P.I=\frac{1}{D^3+D^2D'-DD'^2-D'^3}e^x\cos 2y$$

$$P.I=e^x\frac{1}{(D+1)^3+(D+1)^2D'-(D+1)D'^2-D'^3}\cos 2y$$

$$P.I=e^x realpartof\frac{1}{(D+1)^3+(D+1)^2D'-(D+1)D'^2-D'^3}e^{i2y}$$

$$P.I=e^x realpartof\frac{1}{1+2i+4+8i}e^{i2y}$$

$$=e^x realpartof\frac{1}{5+10i}e^{i2y}=\frac{e^x}{5}R.Pof\frac{1}{1+2i}e^{i2y}$$

$$=\frac{e^x}{5}R.Pof\frac{1(1-2i)}{(1+2i)(1-2i)}e^{i2y}=\frac{e^x}{5}R.Pof\frac{1-2i}{1+4}e^{i2y}$$

$$=\frac{e^x}{25}R.Pof[(1-2i)](\cos 2y+i\sin 2y)$$

$$=\frac{e^x}{25}[\cos 2y+2\sin 2y]$$

The general soln is

$$z=\phi_1(y+x)+\phi_2(y-x)+x\phi_3(y-x)+\frac{e^x}{25}[\cos 2y+2\sin 2y]$$

(vi)Problems based on R.H.S $\sin ax\sin by$ or $\cos ax\cos by$

Solve $(D^2+DD'-6D'^2)Z=y\cos x$

Solution

Given $(D^2+DD'-6D'^2)Z=y\cos x$

The auxiliary equation is $m^2 + m - 6 = 0$

$m = 2, -3$

$C.F = \phi_1(y+2x) + \phi_2(y-3x)$

$$P.I = \frac{1}{D^2 + DD' - DD'^2 - 6D'^2} y\cos x$$

$$P.I = \frac{1}{(D-2D')(D+3D')} y\cos x$$

$$P.I = \frac{1}{(D-2D')}\int (c+3x)\cos x dx \text{ when } y = c+3x$$

$$= \frac{1}{(D-2D')}\left[(c+3x)\sin x - 3\int \sin x dx\right] \text{ when } c = y-3x$$

$$= \frac{1}{(D-2D')}\left[y\sin x + 3\cos x\right]$$

$$= \int [(c-2x)\sin x + 3\cos x]dx \quad \text{ when } y = c-2x$$

$$= [(c-2x)(-\cos x) - (-2)(-\sin x) + 3\sin x] \quad \text{ when } c = y+2x$$

$$= -y\cos x + \sin x$$

Hence the general soln is

$z = .C.F + P.I$

$z = \phi_1(y+2x) + \phi_2(y-3x) - y\cos x + \sin x$

Non homogeneous linear equation

The linear differential equations which are not homogeneous are called non homogeneous linear equation.

$$3\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} + 5\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = 0$$

$f(D, D') = f_1(x, y)$

Its solution Z=C.F+P.I

Problems based on Non-homogeneous linear equation

1. Solve $(D+D'-2)(D+4D,-3)Z=0$

Solution

$(D+D'-2)(D+4D,-3)Z=0$

$[D-(-1)D'-2][D-(-4)D'-3]z=0$

If $(D-m_1D'-c_1)(D-m_2D'-c_2).......(D-m_nD'-c_n)z=0$ **then** $z=e^{c_1x}f_1(y+m_1x)+e^{c_2x}f_2(y+m_2x)+........+e^{c_nx}f_n(y+m_nx)$

$m_1=-1, c_1=2, m_2=-4, c_2=3$

Here $z=e^{2x}f_1(y+(-1)x)+e^{3x}f_2(y+(-)4x)$

$z=e^{2x}f_1(y-x)+e^{3x}f_2(y-4x)$

2. **Solve** $(D-D'-1)(D-D'-2)Z=e^{2x+y}$

Solution

Given $(D-D'-1)(D-D'-2)Z=e^{2x+y}$

To find C.F $(D-D'-1)(D-D'-2)Z=0$

Here $m_1=1, c_1=1, m_2=1, c_2=2$

$C.F=z=e^{x}f_1(y+x)+e^{2x}f_2(y+x)$ **.**

The general soln is $z=e^{x}f_1(y+x)+e^{2x}f_2(y+x)-xe^{2x+y}$

Problems with Zero boundary values-Teperatue or Temperature gradients

Problem

A homogeneous rod of conducting materials of length l has its ends kept at zero temperatue.The temperature at the centre is T and falls uniformly to zero at the ends. Find u(x,t).

Solution

The temperature function u(x,t) satisfies the one dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

From the given problem we get the following boundary and initial conditions

(i) $u(0,t) = 0, \forall t \geq 0$

(ii) $u(l,t) = 0, \forall t \geq 0$

Since the temperature at the centre is T and falls uniformly zero at the two ends, its distribution at t=0 is as given in the figure.

The equation of OB is

$$\frac{x - x_1}{x_2 - x_1} = \frac{x - u_1}{u_2 - u_1}$$

$$\frac{x - 0}{\frac{l}{2} - 0} = \frac{u - 0}{T - 0}$$

$$\frac{2x}{l} = \frac{u}{T}$$

$$u = \frac{2Tx}{l}, 0 \leq x \leq \frac{l}{2}$$

The equation BA is

$$\frac{x - x_1}{x_2 - x_1} = \frac{u - u_1}{u_2 - u_1}$$

$$\frac{x - \frac{l}{2}}{l - \frac{l}{2}} = \frac{u - T}{0 - T}$$

$$\frac{x - \frac{l}{2}}{\frac{l}{2}} = \frac{u - T}{-T}$$

$$\frac{x}{\frac{l}{2}} - 1 = -\frac{u}{T} + 1$$

$$\frac{u}{T} = 1 - \frac{x}{\left(\frac{l}{2}\right)} + 1$$

$$= 2 - \frac{2x}{l} = 2\frac{(l-x)}{l}$$

$$u = \frac{2T(l-x)}{l}, \frac{l}{2} \le x \le l$$

Hence (iii) $u(x,0) = f(x) = \begin{cases} \frac{2Tx}{l} if\ 0 \le x \le \frac{l}{2} \\ \frac{2T(l-x)}{l} if\ \frac{l}{2} \le x \le l \end{cases}$

Now the suitable solution which satisfies the boundary conditions is give by

$$u(x,t) = \left(A\cos px + B\sin px)e^{-\alpha^2 p^2 t}.\right) \text{........(1)}$$

Applying condition (i) in equation (1) we get

$$u(0,t) = (A+0)e^{-\alpha^2 p^2 t}. = 0$$

$$Ae^{-\alpha^2 p^2 t} = 0.$$

Here $e^{-\alpha^2 p^2 t}. \ne 0$ since it is defined for all t

Therefore A=0

Substitute, A=0 in equation (1) we get

$$u(x,t) = (B\sin px)e^{-\alpha^2 p^2 t} \text{........(2)}$$

Now applying condition (ii) in equation (2) we get

$$u(l,t) = (B\sin plx)e^{-\alpha^2 p^2 t} = 0$$

Here $e^{-\alpha^2 p^2 t} \ne 0$, it is defined for all t

$B \ne 0$ If B=0 already A=0 then we get a trivial solution

$\sin pl = 0$ since $\sin n\pi = 0$

$l = n\pi$

$$p=\frac{n\pi}{l}$$

Substitute, $p=\frac{n\pi}{l}$ in equation (2) we get

$$u(x,t)=B\sin\frac{n\pi x}{l}e^{\frac{-\alpha^2 n^2\pi^2 t}{l^2}}$$

$$u(x,t)=B\sin\frac{n\pi x}{l}e^{\frac{-\alpha^2 n^2\pi^2 t}{l^2}}\quad ....(3)$$

The most general solution is

$$u(x,t)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l}e^{\frac{-\alpha^2 n^2\pi^2 t}{l^2}}\quad ......(4)$$

Applying condition (iii) in equation (4) we get

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l}=f(x).....(5)$$

To find B_n expand f(x) in half range Fourier sine series in the interval [0,l]

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}....(6)$$

Where $b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}$

From equation (5) and (6) we get $B_n = b_n$

$$B_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$B_n=\frac{2}{l}\left[\int_0^{l/2}\frac{2Tx}{l}\sin\left(\frac{n\pi x}{l}\right)dx+\int_{l/2}^{l}\frac{2T(l-x)}{l}\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$=\frac{4T}{l^2}\left[\int_0^{l/2} x\sin\left(\frac{n\pi x}{l}\right)dx+\int_{l/2}^{l}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$=\frac{4T}{l^2}x\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]-(1)\left[\left[\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]\right]_0^{l/2}$$

$$+\frac{4T}{l^2}[(l-x)]\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]-(-1)\left[\left[\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]\right]_{l/2}^{l}$$

$$=\frac{4T}{l^2}\left[-x\left(\frac{l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right)+\left(\frac{l}{n\pi}\right)\sin\left(\frac{n\pi x}{l}\right)\right]_0^{l/2}$$

$$+\frac{4T}{l^2}\left[-(l-x)\left(\frac{l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right)-\left(\frac{l}{n\pi}\right)^2\sin\left(\frac{n\pi x}{l}\right)\right]_0^{l/2}$$

$$=\frac{4T}{l^2}\left[-\left(\frac{l^2}{2n\pi}\right)\cos\left(\frac{n\pi}{2}\right)+\left(\frac{l^2}{2n\pi}\right)\cos\left(\frac{n\pi}{2}\right)+\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]_0^{l/2}$$

$$=\frac{4T}{n^2l^2}\sin\left(\frac{n\pi}{2}\right)$$

Substitute B_n values in equation (4) we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8T}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$= \frac{8T}{n^2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}$$

2. A rod 30 cm long its ends A and B kept at 20^o and 80^o respectively until steady state conditions prevail. The temperature at each and is then suddenly reduced to $0^o C$ and kept so. Find the resulting temperature function $u(x,t) = 0$ taking x=0 at A.

Solution

The temperature function $u(x,t)$ is the solution of the one dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} ....(A)$ when the steady state condition prevails $\frac{\partial u}{\partial t} = 0$ and hence we get

$\frac{\partial^2 u}{\partial x^2} = 0$. Therefore (A) reduces to $\frac{d^2 u}{dx^2} = 0$, on integration

$u(x) = ax + b..........(B)$

When x=0, we get $u(0) = b = 20$

When x=30, we get $u(30) = 30a + b$

$80 = 30a + 20, a = 2$

Thus $u(x,0) = f(x) = 2x + 20$ by (B)

Hence Boundary and initial condition are

(i) $u(0,t) = 0, \forall t \ge 0$

(ii) $u(30,t) = 0, \forall t \ge 0$

(iii) $u(x,0) = f(x) = 2x + 20$

Now the suitable solution which statisfies our boundary conditions is given by

$$u(x,t) = \left(A\cos px + B\sin px)e^{-\alpha^2 p^2 t}.\right)........(1)$$

Applying condition (i) in equation (1) we get

$$u(0,t) = (A+0)e^{-\alpha^2 p^2 t}. = 0$$

$$Ae^{-\alpha^2 p^2 t} = 0.$$

Here $e^{-\alpha^2 p^2 t}. \neq 0$ since it is defined for all t

Therefore A=0

Substitute, A=0 in equation (1) we get

$$u(x,t) = (B\sin px)e^{-\alpha^2 p^2 t}........(2)$$

Now applying condition (ii) in equation (2) we get

$$u(30,t) = (B\sin 30p)e^{-\alpha^2 p^2 t} = 0$$

Here $e^{-\alpha^2 p^2 t} \neq 0$, it is defined for all t

$B \neq 0$ If B=0 already A=0 then we get a trivial solution

$$\begin{aligned} \sin 30p &= 0 \\ \sin 30p &= \sin n\pi \end{aligned} \quad \text{since } \sin n\pi = 0$$

$$30p = n\pi$$

$$p = \frac{n\pi}{30}$$

Substitute, $p = \dfrac{n\pi}{30}$ in equation (2) we get

$$u(x,t) = B\sin\frac{n\pi x}{30}e^{\frac{-\alpha^2 n^2 \pi^2 t}{900}} ....(3)$$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30}e^{\frac{-\alpha^2 n^2 \pi^2 t}{900}} ......(4)$$

Applying condition (iii) in equation (4) we get

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{30}=2x+20.....(5)$$

To find B_n expand $2x+20$ in half range Fourier sine series in the interval [0,30]

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{30}....(6)$$

Where $b_n=\frac{2}{30}\int_0^l f(x)\sin\frac{n\pi x}{30}dx$

From equation (5) and (6) we get $B_n = b_n$

$$B_n=\frac{2}{30}\int_0^{30}(2x+20)\sin\frac{n\pi x}{30}dx$$

$$B_n=\frac{1}{15}\left[(2x+20)\left(\frac{-\cos\frac{n\pi x}{30}}{\left(\frac{n\pi}{30}\right)}\right)-(2)\left[\frac{\frac{\sin\frac{n\pi x}{30}}{2}}{\left(\frac{n\pi}{30}\right)}\right]\right]_0^{30}$$

$$B_n=\frac{1}{15}\left[-(2x+20)\left(\frac{30}{n\pi}\right)\cos\frac{n\pi x}{30}+(2)\left(\frac{30}{n\pi}\right)^2\sin\frac{nx\pi}{30}\right]_0^{30}$$

$$B_n=\frac{1}{15}\left[-\left(\frac{2400}{n\pi}\cos n\pi+0\right)-\left(\frac{-600}{n\pi}\right)-0\right]$$

$$=\frac{1}{15}\left[-\frac{2400}{n\pi}\cos n\pi+\left(\frac{600}{n\pi}\right)\right]$$

$$=\frac{600}{15n\pi}[-4\cos n\pi+1]$$

$$=\frac{40}{n\pi}[1-4\cos n\pi]$$

$$=\frac{40}{n\pi}[1-4(-1)^n]$$

Substitute the value of B_n in equation (4) we get

$$u(x,t) = \frac{40}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-4(-1)^n}{n}\right]\sin\frac{n\pi x}{30}\, e^{\frac{-\alpha^2 n^2 \pi^2 t}{900}}.$$

UNIT-II
FOURIER SERIES

1.0 INTRODUCTION

Fourier Series is named after the French mathematician cum physicist Jean- Baptiste Joseph fourier (1768-1830).He introduced Fourier Series in 1822,while he was investigating the problem of heat conduction. The series of sines and cosines is known after him.

Use of Fourier Series:

Fourier Series are particularly suitable for expansion of periodic functions. We come across many periodic functions in voltage, current, flux density, applied force, potential and electromagnetic force in electricity. Hence Fourier Series are very useful in electrical engineering problems.

Periodic Function:

A function f(x) is said to be periodic if and only if f(x+p) = f(x) is true for some value of p and every value of x. The smallest value of p for which this equation is true for every value of x will be called the periodic of the function f(x).

Fourier Series:

Here we will express a non-sinusoidal periodic function into a fundamental and its harmonics, a series of sines and cosines of an angle and its multiple of the form

$$\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+....+a_n\cos nx+.....+b_1\sin x+b_2\sin 2x+....+b_n\sin nx+....$$

$=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ is called the fourier series, where $a_0,a_1,......a_n$, $b_1,b_2,....b_n$ are constants.

1.1(a) PROBLEMS UNDER THE INTERVAL (0,2п)

Write the formula for finding Euler's constant of a fourier series in $(0,2\pi)$.

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

where $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx$ ....(1)

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nxdx \text{ ....(2)}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nxdx \text{ ....(3)}$$

Formulas (1)),(2),|(3) are known as the Euler's formula.

Explain Dirichlet's condition

Any function f(x) can be developed as a fourier series $\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ where a_0,a_n,b_n are constants, provided.

(i) f(x) is periodic,single-valued and finite

(ii) f(x) has a finite number of finite discontinuities in any one period and has no infinite discontinuity.

(iii) f(x) has the most a finite number of maxima and minima.

Problem-1:

If $f(x)=\frac{1}{2}(\pi - x)$,find the fourier series of period 2π in the interval (0,2π).Hence deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+....=\frac{\pi}{4}$.

solution:

Let the required Fourier Series be $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ ……….(1)

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)dx$

$=\frac{1}{2\pi}\int_0^{2\pi}(\pi - x)dx$

$=\frac{1}{2\pi}\left[\pi x-\frac{x^2}{2}\right]_0^{2\pi}$

$=\frac{1}{2\pi}\left[\left(2\pi^2-\frac{4\pi^2}{2}\right)-(0-0)\right]$

$=\frac{1}{2\pi}\left[\left(2\pi^2-2\pi^2\right)\right]$

$=\frac{1}{2\pi}[0]$

$=0$

$\therefore a_0 = 0$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx$

$=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\cos nxdx$

$=\frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\cos nxdx$

$=\frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right)-(-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$

$=\frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right)-\left(\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$

$$=\frac{1}{2\pi}\left[\left(0-\frac{1}{n^2}\right)-\left(0-\frac{1}{n^2}\right)\right]$$

$$=\frac{1}{2\pi}\left[\left(-\frac{1}{n^2}\right)+\left(\frac{1}{n^2}\right)\right]$$

$$=\frac{1}{2\pi}[0]$$

$$=0$$

$$\therefore a_n=0$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$

$$=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\sin nx dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi}(\pi-x)\sin nx dx$$

$$=\frac{1}{2\pi}\left[((\pi-x)(\frac{-\cos nx}{n})-(-1)\left(\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[(-(\pi-x)(\frac{\cos nx}{n})-\left(\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[(\frac{\pi}{n}-0)-\left(\frac{-\pi}{n}\right)\right]$$

$$=\frac{1}{2\pi}\left[(\frac{\pi}{n})+\left(\frac{\pi}{n}\right)\right]$$

$$=\frac{1}{2\pi}\left[\frac{2\pi}{n}\right]$$

$$=\frac{1}{n}$$

$$\therefore b_n=\frac{1}{n}$$

substitute $a_n=0$ & $b_n=\frac{1}{n}$ in (1),we get

$$f(x)=0+0+\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$f(x)=\sum_{n=1}^{\infty}\frac{1}{n}\sin nx \text{ ....(2)}$$

put $x=\frac{\pi}{2}$,here $x=\frac{\pi}{2}$ is a point of continuity

$\therefore f(x)) = \frac{1}{2}\left(\pi - \frac{\pi}{2}\right)$

$= \frac{\pi}{4}$

$(2) \Rightarrow \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2}$

$\frac{\pi}{4} = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7} .....$

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} .....$

(i.e) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} ..... = \frac{\pi}{4}$.

1.1(b) PROBLEMS UNDER THE INTERVAL (0,2l)

Write the formula for fourier constant for f(x) in (0,2l).

The fourier expansion for f(x) in the interval 0 < x < 2l is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2n\pi x}{3}) + \sum_{n=1}^{\infty} b_n \sin(\frac{2n\pi x}{3})$$

where $a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$ ………..(1)

$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$ ……….(2)

$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$ ……….(3)

Problem-2:

Find the fourier series of the function f(x) = 2x- x^2 for $0 < x < 3$ and f(x+3) = f(x).

solution:

given $2l = 3$

$l = \frac{3}{2}$

Let the required Fourier Series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2n\pi x}{3}) + \sum_{n=1}^{\infty} b_n \sin(\frac{2n\pi x}{3}) \;\text{………}(1)$$

where $a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$

$= \frac{2}{3}\int_0^{3} (2x - x^2)dx$

$$=\frac{2}{3}\left[\frac{2x^2}{2}-\frac{x^3}{3}\right]_0^3$$

$$=\frac{2}{3}(0)$$

$$=0$$

$$\therefore a_0=0$$

$$a_n=\frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{3}\int_0^3 (2x-x^2)\cos(\frac{2n\pi x}{3})dx$$

$$=\frac{2}{3}\left[(2x-x^2)\left(\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)-(2-2x)\left(-\frac{\cos\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right)+(-2)\left(-\frac{\sin\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}\right)\right]_0^3$$

$$=\frac{2}{3}\left[\left(\ (2(3)-3^2)\left(\frac{\sin\frac{2n\pi(3)}{3}}{\frac{2n\pi}{3}}\right)-(2-2(3))\left(-\frac{\cos\frac{2n\pi(3)}{3}}{\frac{4n^2\pi^2}{9}}\right)+(-2)\left(-\frac{\sin\frac{2n\pi(3)}{3}}{\frac{8n^3\pi^3}{27}}\right)\ \right)-\left(0-(2)\frac{1}{\frac{4n^2\pi^2}{9}}+0\right)\right]$$

$$=\frac{2}{3}\left[0-4(\frac{9}{4n^2\pi^2})+0)-\left(0+(2)\frac{9}{4n^2\pi^2}+0\right)\right]$$

$$=\frac{2}{3}\left[(-\frac{9}{n^2\pi^2})-\left(\frac{9}{2n^2\pi^2}\right)\right]$$

$$=\frac{2}{3}\left[(-\frac{9}{n^2\pi^2})\left(1+\frac{1}{2}\right)\right]$$

$$=\frac{2}{3}\left[(-\frac{9}{n^2\pi^2})\left(\frac{3}{2}\right)\right]$$

$$=-\frac{9}{n^2\pi^2}$$

$$\therefore a_n=-\frac{9}{n^2\pi^2}$$

$$b_n=\frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{3}\int_0^3 (2x-x^2)\sin(\frac{2n\pi x}{3})dx$$

$$=\frac{2}{3}\left[\left(2x-x^2\right)\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)-(2-2x)\left(-\frac{\sin\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right)+(-2)\left(-\frac{\cos\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}\right)\right]_0^3$$

$$=\frac{2}{3}\left[(-\left(2(3)-3^2\right)\left(\frac{\cos\frac{2n\pi(3)}{3}}{\frac{2n\pi}{3}}\right)-(2-2(3))\left(-\frac{\sin\frac{2n\pi(3)}{3}}{\frac{4n^2\pi^2}{9}}\right)+(-2)\left(-\frac{\cos\frac{2n\pi(3)}{3}}{\frac{8n^3\pi^3}{27}}\right))-\left(0+0+\frac{-27}{4n^3\pi^3}\right)\right]$$

$$=\frac{2}{3}\left[\left(\frac{9}{2n\pi}-\frac{27}{4n^3\pi^3}\right)+\left(\frac{27}{4n^3\pi^3}\right)\right] \qquad \because \cos 2n\pi=1$$

$$=\frac{2}{3}\left[\frac{9}{2n\pi}\right]$$

$$=\frac{3}{n\pi}$$

$$b_n=\frac{3}{n\pi}$$

$$(1)\Rightarrow\quad f(x)=0+\sum_{n=1}^{\infty}-\frac{9}{n^2\pi^2}\cos\frac{2n\pi x}{3}+\sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\frac{2n\pi x}{3}$$

$$f(x)=\frac{-9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{3}+\frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{3}$$

1.2 ODD AND EVEN FUNCTIONS

Certain functions defined in symmetric ranges of the form $(-\pi,\pi)$ and $(-l,l)$ can be classified as odd and even functions.

Definition

f(x) is said to be an even functions of x in $(-l,l)$ if f(-x) = f(x) .Geometrically the graph of even function will be symmetrical w.r.t Y axis.

example: xsinx,cosx

Definition

f(x) is said to be an odd functions of x in $(-l,l)$ if f(-x) = - f(x) .Geometrically the graph of an odd function will be symmetrical about the origin.

example: sinx,xcosx.

1.2(a)PROBLEMS UNDER $(-\pi,\pi)$

What are the values of the Fourier constants when an even function f(x) is expanded in a fourier series in $(-\pi,\pi)$?

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx$$

$b_n = 0$

The expansion will be of the form $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

What are the values of the fourier constants when an odd function f(x) is expanded in a fourier series in $(-\pi,\pi)$?

solution:

$a_0 = 0$, $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

The expansion will be of the form $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$.

What are the values of the fourier constants when f(x) is neither even nor odd in a fourier series in $(-\pi,\pi)$?

Let the fourier series for f(x) in $(-\pi,\pi)$ be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$ ……….(1)

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$ ……….(2)

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$ ………..(3)

Formulas (1)),(2),|(3) are known as the Euler's formula.

Problem-3:

Find the fourier expansion of f(x) = x in $(-\pi,\pi)$.

solution:

Let f(x) = x,f(-x) = -x = - f(x)

$\therefore$ f(x) is an odd function in $(-\pi,\pi)$

hence $a_0 = 0$, $a_n = 0$ for all n > 0

Let the required fourier series be $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx dx$$

$$b_n = \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-x\left(\frac{\cos nx}{n}\right)+\left(\frac{\sin nx}{n}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{-\pi(-1)^n}{n}+0\right)-(0+0)\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{-\pi(-1)^n}{n}\right)\right]$$

$$= \frac{2(-1)^{n+1}}{n}$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$f(x) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$$

$$\therefore f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\ .$$

Problem-4:

Find the fourier series for f(x) = x^2 in $(-\pi,\pi)$ and deduce that (i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.......=\frac{\pi^2}{6}$

(ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-.......=\frac{\pi^2}{12}$ (iii) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.......=\frac{\pi^2}{8}$

solution:

Let f(x) = x^2, f(-x) = $(-x)^2$ = x^2 = f(x)

$\therefore$ f(x) is an even function in $(-\pi,\pi)$

hence $b_n = 0$

Let the required fourier series be $f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$ where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3}-0\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3}\right]$$

$$= \frac{2\pi^2}{3}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx$$

$$a_n = \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) + (2x)\left(\frac{\cos nx}{n^2}\right) - 2\left(\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 + \left(\frac{2\pi(-1)^n}{n^2}\right) - 0\right) - (0 + 0 - 0)\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right]$$

$$= \frac{4}{n^2}(-1)^n$$

$$a_n = \frac{4}{n^2}(-1)^n$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \frac{2\pi^2}{3(2)} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx \quad \ldots\ldots\ldots\ldots\ldots..(1)$$

Deduction:

(i)put $x = \pi$ in f(x) we get

f(x) = π^2

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos n\pi$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \because (-1)^n(-1)^n = (-1)^{2n} = 1$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\right)$$

$$\frac{3\pi^2-\pi^2}{3}=4\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.......\right)$$

$$\frac{2\pi^2}{3}=4\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.......\right)$$

$$\frac{\pi^2}{3}=2\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.......\right)$$

$$\frac{\pi^2}{6}=\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.......\right)$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.......=\frac{\pi^2}{6} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

(ii) put x = 0 in f(x) we get f(x) = 0
substitute f(x) = 0 in (1)

$$0=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n(0)$$

$$0=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{3}=4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12}=\frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+......$$

$$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}+......$$

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}+......=\frac{\pi^2}{12} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

(iii) add (2) & (3) we get

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.......\right)=\frac{\pi^2}{6}+\frac{\pi^2}{12}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+......=\frac{1}{2}\left(\frac{2\pi^2+\pi^2}{12}\right)$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+......=\frac{1}{2}\left(\frac{3\pi^2}{12}\right)$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+......=\frac{\pi^2}{8}\ .$$

Problem-5:

Obtain the Fourier series for f(x) = 1+x+x^2 in $\left(-\pi,\pi\right)$.deduce that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.......=\frac{\pi^2}{6}$.

solution:

Let f(x) = 1+x+x^2, f(-x) = 1+(-x)+(-x^2)= 1-x+x^2

here $f(-x)\neq f(x)$ and $f(-x)\neq -f(x)$

Hence the given function is neither even nor odd.

Let the Fourier series for f(x) in $\left(-\pi,\pi\right)$ be $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ ....(1)

where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(1+x+x^2)dx$$

$$=\frac{1}{\pi}\left(x+\frac{x^2}{2}+\frac{x^3}{3}\right)_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[\left(\pi+\frac{\pi^2}{2}+\frac{\pi^3}{3}\right)-\left(-\pi+\frac{\pi^2}{2}-\frac{\pi^3}{3}\right)\right]$$

$$=\frac{1}{\pi}\left[\pi+\frac{\pi^2}{2}+\frac{\pi^3}{3}+\pi-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right]$$

$$=\frac{1}{\pi}\left[2\pi+\frac{2\pi^3}{3}\right]$$

$$=\frac{1}{\pi}\left[\frac{6\pi+2\pi^3}{3}\right]$$

$$=\frac{1}{\pi}\left[\frac{6\pi+2\pi^3}{3}\right]$$

$$=\frac{1}{3\pi}(2\pi)\left[3+\pi^2\right]$$

$$=\frac{2}{3}\left[3+\pi^2\right]$$

$$a_0=\frac{2}{3}\left[3+\pi^2\right]$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(1+x+x^2)\cos nx dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos nxdx+\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos nxdx+\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nxdx$$

$$=\frac{2}{\pi}\int_{0}^{\pi}\cos nxdx+0+\frac{2}{\pi}\int_{0}^{\pi}x^2\cos nxdx \qquad \because \text{ xcosnx is a odd function, } \therefore \int_{-\pi}^{\pi}x\cos nxdx=0$$

$$=\frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}+\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-2x\left(\frac{(-\cos nx)}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}(0-0)+\frac{2}{\pi}\left[\left(0+\frac{2\pi(-1)^n}{n^2}+0-(0+0-0)\right)\right]$$

$$=\frac{2}{\pi}\left[\left(\frac{2\pi(-1)^n}{n^2}\right)\right]$$

$$=\frac{4(-1)^n}{n^2}$$

$$a_n=\frac{4(-1)^n}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(1+x+x^2)\sin nxdx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin nxdx+\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nxdx+\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin nxdx$$

$$=0+\frac{2}{\pi}\int_{0}^{\pi}x\sin nxdx+0 \qquad \because \text{ sinnx , xsinnx is a odd function } \therefore \int_{-\pi}^{\pi}x^2\sin nxdx=0, \int_{-\pi}^{\pi}\sin nxdx=0$$

$$=\frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)-(1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[-x\left(\frac{\cos nx}{n}\right)+(1)\left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left(-\pi\left(\frac{\cos n\pi}{n}\right)+(0)-(-0+0)\right)\right]$$

$$=\frac{2}{\pi}\left(\frac{-\pi(-1)^n}{n}\right)$$

$$=\frac{-2(-1)^n}{n}$$

$$b_n=\frac{-2(-1)^n}{n}$$

substitute $a_0 = \frac{2}{3}\left[3+\pi^2\right], a_n = \frac{4(-1)^n}{n^2}, b_n = \frac{-2(-1)^n}{n}$ in (1)

$$f(x) = \frac{2}{2(3)}\left[3+\pi^2\right] + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{-2(-1)^n}{n}\sin nx$$

$$f(x) = \frac{1}{3}\left[3+\pi^2\right] + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx \,\ldots\ldots\ldots(2)$$

put x = π in f(x) we get $f(x) = \frac{f(-\pi)+f(\pi)}{2}$

$$f(x) = \frac{(1-\pi+\pi^2)+(1+\pi+\pi^2)}{2}$$

$$= \frac{2+2\pi^2}{2}$$

$$= 1+\pi^2$$

$$f(x) = 1+\pi^2$$

$$(2) \Rightarrow 1+\pi^2 = \frac{1}{3}\left[3+\pi^2\right] + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi$$

$$1+\pi^2 = \frac{1}{3}\left[3+\pi^2\right] + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi$$

$$1+\pi^2 = \frac{1}{3}\left[3+\pi^2\right] + 4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2}$$

$$3(1+\pi^2) = \left[3+\pi^2\right] + 12\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$3+3\pi^2 = 3+\pi^2 + 12\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots.\right)$$

$$3\pi^2 = \pi^2 + 12\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots.\right)$$

$$2\pi^2 = 12\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots.\right)$$

$$\pi^2 = 6\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots.\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots.$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots. = \frac{\pi^2}{6}.$$

1.2(b)PROBLEMS UNDER $(-l,l)$

What are the values of the fourier constants when an even function f(x) is expanded in a fourier series in $(-l,l)$?

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = 0$$

The expansion will be of the form $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

What are the values of the fourier constants when an odd function f(x) is expanded in a fourier series in $(-l,l)$?

$a_0 = 0$, $a_n = 0$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

The expansion will be of the form $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$.

What are the values of the fourier constants when f(x) is neither even nor odd in a Fourier series in $(-l,l)$?

Let the fourier series for f(x) in $(-l,l)$ be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

<u>Problem-6:</u>

If f(x) = x is defined in $(-l,l)$ with period $2l$,find the fourier expansion of f(x).

<u>solution:</u>

f(x) = x

f(-x) = -x = -f(x)

$\therefore$ f(x) is an odd function.

Hence $a_0 = 0$, $a_n = 0$

Let the Fourier series be $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ ……….(1)where $b_n = \frac{2}{l}\int_0^{\pi} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$=\frac{2}{l}\int_0^l x\sin\left(\frac{n\pi x}{l}\right)dx$$

$$=\frac{2}{l}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$=\frac{2}{l}\left[-x\left(\frac{l}{n\pi}\right)\cos\frac{n\pi x}{l}+\left(\frac{l^2}{n^2\pi^2}\right)\sin\frac{n\pi x}{l}\right]_0^l$$

$$=\frac{2}{l}\left[\left(\left(\frac{-l^2}{n\pi}\right)\cos n\pi+\left(\frac{l^2}{n^2\pi^2}\right)\sin n\pi\right)-(0+0)\right]$$

$$=\frac{2}{l}\left[\left(\frac{-l^2}{n\pi}\right)\cos n\pi\right]$$

$$=\frac{-2l(-1)^n}{n\pi}$$

$$b_n=\frac{-2l(-1)^n}{n\pi}$$

substitute $b_n=\frac{-2l(-1)^n}{n\pi}$ in (1)

The Fourier series is $f(x)=\sum_{n=1}^{\infty}\frac{-2l(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{l}\right)$

$$f(x)=\frac{-2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{l}\right).$$

Problem-7:

Find the Fourier series of the following function $f(x)=\begin{cases}0,-2<x<-1\\ k,-1<x<1\\ 0,1<x<2\end{cases}$

solution:

here $2l=4$

$l=2$

f(x) is an even function. hence $b_n=0$

Let the Fourier series be $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)$…….(1)where $a_0=\frac{2}{l}\int_0^l f(x)dx$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$a_0 = \frac{2}{2}\int_0^1 k dx$$
$$= k(x)_0^1$$
$$= k(1-0) = \text{k}$$
$$a_0 = k$$
$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$
$$= \frac{2}{2}\int_0^1 k\cos\left(\frac{n\pi x}{2}\right)dx$$
$$= k\int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx$$
$$= k\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)_0^1$$
$$= k\left(\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}} - 0\right)$$
$$= k\left(\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}}\right)$$
$$= k\frac{2}{n\pi}\sin\frac{n\pi}{2}$$
$$a_n = k\frac{2}{n\pi}\sin\frac{n\pi}{2}$$
substitute $a_0 = k$, $a_n = k\frac{2}{n\pi}\sin\frac{n\pi}{2}$ in (1)
$$f(x) = \frac{k}{2} + \sum_{n=1}^{\infty} k\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\left(\frac{n\pi x}{2}\right)$$
$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\cos\left(\frac{n\pi x}{2}\right).$$
Problem-8:

Expand $f(x) = e^{-x}$ as a Fourier series in (-1,1).

solution:

$f(x) = e^{-x}$

$f(-x) = e^{x}$

$\therefore f(x) \neq f(-x) \;\&\; \therefore f(-x) \neq -f(x)$

$\therefore$ f(x) is neither even nor odd.

Let the Fourier series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ ……….(1)

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$$= \frac{1}{1}\int_{-1}^{1} e^{-x}dx$$

$$= \frac{2}{1}\int_{0}^{1} e^{-x}dx$$

$$= 2\left[-e^{-x}\right]_0^1$$

$$= 2\left[-e^{-1} + e^{0}\right]$$

$$= 2\left[\frac{-1}{e} + 1\right]$$

$$= 2\left[\frac{-1+e}{e}\right] = 2\sinh 1$$

$a_0 = 2\sinh 1$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= 1\int_{-1}^{1} e^{-x}\cos\left(\frac{n\pi x}{1}\right)dx \qquad = \left[\frac{e^{-x}}{1+n^2\pi^2}\left(-\cos n\pi x + n\pi\sin n\pi x\right)\right]_{-1}^{1}$$

$$= \frac{1}{1+n^2\pi^2}\left[e^{-x}\left(-\cos n\pi x + n\pi\sin n\pi x\right)\right]_{-1}^{1}$$

$$= \frac{1}{1+n^2\pi^2}\left[-e^{-1}(-1)^n + e^{1}(-1)^n\right]$$

$$= \frac{(-1)^n}{1+n^2\pi^2}\left[-e^{-1} + e^{1}\right]$$

$$= \frac{2(-1)^n}{1+n^2\pi^2}\sinh 1$$

$$a_n = \frac{2(-1)^n}{1+n^2\pi^2}\sinh 1$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= 1\int_{-1}^{1} e^{-x}\sin\left(\frac{n\pi x}{1}\right)dx$$

$$= \left[\frac{e^{-x}}{1+n^2\pi^2}\left(-\sin n\pi x - n\pi\cos n\pi x\right) \right]_{-1}^{1}$$

$$= \frac{-1}{1+n^2\pi^2}\left[e^{-x}\left(\sin n\pi x + n\pi\cos n\pi x\right)\right]_{-1}^{1}$$

$$= \frac{-1}{1+n^2\pi^2}\left[n\pi e^{-1}(-1)^n - n\pi e^{1}(-1)^n\right]$$

$$= \frac{(-1)^n n\pi}{1+n^2\pi^2}\left[-e^{-1}+e^{1}\right]$$

$$= \frac{2(-1)^n n\pi}{1+n^2\pi^2}\sinh 1$$

$$b_n = \frac{2(-1)^n n\pi}{1+n^2\pi^2}\sinh 1$$

substitute $a_{0,}a_n,b_n$ in (1)

$$f(x) = \frac{2\sinh 1}{2} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{1+n^2\pi^2}\sinh 1\cos n\pi x + \sum_{n=1}^{\infty}\frac{2(-1)^n n\pi}{1+n^2\pi^2}\sinh 1\sin n\pi x$$

$$f(x) = \sinh 1 + 2\sinh 1\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2\pi^2}\cos n\pi x + 2\sinh 1\sum_{n=1}^{\infty}\frac{(-1)^n n\pi}{1+n^2\pi^2}\sin n\pi x$$

$$f(x) = \sinh 1 + 2\sinh 1\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2\pi^2}\cos n\pi x + 2\sinh 1\sum_{n=1}^{\infty}\frac{(-1)^n n\pi}{1+n^2\pi^2}\sin n\pi x$$

$$f(x) = \sinh 1\left[1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2\pi^2}\cos n\pi x + 2\sum_{n=1}^{\infty}\frac{(-1)^n n\pi}{1+n^2\pi^2}\sin n\pi x\right].$$

<u>1.3.(a) HALF RANGE SINE SERIES</u>

<u>sine series:</u>

To expand f(x) as a sine series in $(0,\pi)$ or $(0,l)$ we extend the function reflecting it in the origin, so that f(-x) = -f(x)

Write the formula for Fourier constants to expand f(x) as a sine series in $(0,\pi)$.

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$,where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

Write the formula for Fourier constants to expand f(x) as a sine series in $(0,l)$.

$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$,where $b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}dx$.

<u>Problem-9:</u>

Expand the function f(x) = x , $(0,\pi)$ in the Fourier sine series.

<u>solution:</u>

given (x) = x , $(0,\pi)$

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ ………..(1) where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx dx$$

$$b_n = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)-(1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\left(\pi\left(\frac{-\cos n\pi}{n}\right)-(1)\left(\frac{-\sin n\pi}{n^2}\right)\right)-(0+0)\right]$$

$$b_n = \frac{2}{\pi}\left(\frac{-\pi(-1)^n}{n}\right)$$

$$b_n = \frac{-2(-1)^n}{n}$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty}\frac{-2(-1)^n}{n}\sin nx$$

$$f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

Problem-10:

Find the Half range Fourier sine series for f(x) = x in $(0,l)$.

solution:

Let the required Fourier s series be $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ …….(1)

where $b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}dx$

$$b_n = \frac{2}{l}\int_0^{l} x\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l}$$

$$= \frac{2}{l}\left[-x\left(\frac{l}{n\pi}\right)\left(\cos\frac{n\pi x}{l}\right)+\left(\frac{l^2}{n^2\pi^2}\right)\left(\sin\frac{n\pi x}{l}\right)\right]_0^{l}$$

$$= \frac{2}{l}\left[-l\left(\frac{l}{n\pi}\right)\left(\cos\frac{n\pi l}{l}\right)+\left(\frac{l^2}{n^2\pi^2}\right)\left(\sin\frac{n\pi l}{l}\right)-(0+0)\right]$$

$$= \frac{2}{l}\left[\left(\frac{-l^2}{n\pi}\right)\cos n\pi+\left(\frac{l^2}{n^2\pi^2}\right)(\sin n\pi)\right]$$

$$= \frac{2}{l}\left[\left(\frac{-l^2}{n\pi}\right)\cos n\pi+0\right] \qquad \because \sin n\pi = o$$

$= \frac{2}{l}\left(\frac{-l^2}{n\pi}\right)(-1)^n$

$= \frac{-2l(-1)^n}{n\pi}$

$b_n = \frac{-2l(-1)^n}{n\pi}$

substitute $b_n = \frac{-2l(-1)^n}{n\pi}$ in (1) we get

$f(x) = \sum_{n=1}^{\infty} \frac{-2l(-1)^n}{n\pi} \sin\frac{n\pi x}{l}$

$f(x) = \frac{-2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi x}{l}$.

1.3.(b) HALF RANGE COSINE SERIES

Cosine series:

To expand f(x) as a cosine series in $(0,\pi)$ or $(0,l)$ we extend the function reflecting it in the y-axis, so that f(-x) = f(x)

Write the formula for Fourier constants to expand f(x) as a cosine series in $(0,\pi)$.

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$ $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$

Write the formula for Fourier constants to expand f(x) as a cosine series in $(0,l)$.

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ where $a_0 = \frac{2}{l}\int_0^{l} f(x)dx$ $a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l} dx$.

Problem-11:

Find a cosine series for f(x) = x in $(0,1)$

solution:

The cosine series of f(x) in (0,1) is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ …………(1)

$a_0 = \frac{2}{1}\int_0^1 f(x)dx$

$= 2\int_0^1 x dx$

$= 2(x^2)_0^1$

$=2(\frac{1}{2})$

$= 1$

$a_0 = 1$ …………(2)

$a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x dx$

$$= 2\int_0^1 x\cos n\pi x dx$$

$$= 2\left[x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\frac{-\cos n\pi x}{n^2\pi^2}\right]_0^1$$

$$= 2\left[\frac{x\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_0^1$$

$$= 2\left[\left(\frac{\sin n\pi}{n\pi} + \frac{\cos n\pi}{n^2\pi^2}\right) - \left(0 + \frac{\cos 0}{n^2\pi^2}\right)\right]$$

$$= 2\left[\left(\frac{(-1)^n}{n^2\pi^2}\right) - \left(\frac{1}{n^2\pi^2}\right)\right]$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} 0, when\ 'n'\ is even \\ \dfrac{-4}{n^2\pi^2}, when\ 'n'\ is odd \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

substituting (2) &(3) in (1), we get

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5}^{\infty} \frac{-4}{n^2\pi^2}\cos n\pi x$$

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1,3,5}^{\infty} \frac{1}{n^2}\cos n\pi x\ .$$

Problem-12:

Obtain a Fourier cosine series for $f(x) = \begin{cases} \cos x, 0 < x < \dfrac{\pi}{2} \\ 0, \dfrac{\pi}{2} < x < \pi \end{cases}$

solution:

The half range cosine series $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ ……….(1)

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} f(x)dx + \int_{\frac{\pi}{2}}^{\pi} f(x)dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\pi} 0 dx\right]$$

$$= \frac{2}{\pi}\left[\sin x\right]_0^{\pi/2}$$

$$= \frac{2}{\pi} \sin \frac{\pi}{2}$$

$$= \frac{2}{\pi}$$

$$a_0 = \frac{2}{\pi} \ \ldots\ldots\ldots.(2)$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} f(x) \cos nx dx + \int_{\frac{\pi}{2}}^{\pi} f(x) \cos nx dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} \cos x \cos nx dx + \int_{\frac{\pi}{2}}^{\pi} 0 \cos nx dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} \cos x \cos nx dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\frac{\pi}{2}} \{ \cos(n+1)x + \cos(n-1)x \} dx \right]$$

$$= \frac{1}{\pi} \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi} \left[\left\{ \frac{\sin(n+1)\frac{\pi}{2}}{(n+1)} + \frac{\sin(n-1)\frac{\pi}{2}}{(n-1)} \right\} - \{0+0\} \right]$$

$$= \frac{1}{\pi} \left\{ \frac{\sin(n+1)\frac{\pi}{2}}{(n+1)} + \frac{\sin(n-1)\frac{\pi}{2}}{(n-1)} \right\}$$

$$= \frac{1}{\pi} \left\{ \left[\frac{\sin\frac{n\pi}{2} \cos\frac{\pi}{2} + \cos\frac{n\pi}{2} \sin\frac{\pi}{2}}{n+1} \right] + \left[\frac{\sin\frac{n\pi}{2} \cos\frac{\pi}{2} - \cos\frac{n\pi}{2} \sin\frac{\pi}{2}}{n-1} \right] \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{\cos\frac{n\pi}{2}\sin\frac{\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}\sin\frac{\pi}{2}}{n-1} \right\}$$

$$= \frac{\cos\frac{n\pi}{2}}{\pi}\left\{ \frac{1}{n+1} - \frac{1}{n-1} \right\}$$

$$= \frac{\cos\frac{n\pi}{2}}{\pi}\left\{ \frac{n-1-n-1}{n^2-1} \right\}$$

$$= \frac{\cos\frac{n\pi}{2}}{\pi}\left\{ \frac{-2}{n^2-1} \right\}$$

$$a_n = \frac{-2\cos\frac{n\pi}{2}}{\pi(n^2-1)} \text{ provided } n \neq 1 \quad \text{……….(3)}$$

when n = 1,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} f(x)\cos x dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\cos x dx \right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x.\cos x dx + \int_{\frac{\pi}{2}}^{\pi} 0\cos x dx \right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos^2 x dx \right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2} dx \right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}x + \frac{\sin 2x}{4} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left\{ \left[\frac{\pi}{4} + \frac{\sin\pi}{4} \right] - \{0+0\} \right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{4}\right\}$$

$$= \frac{1}{2}$$

$$a_1 = \frac{1}{2} \ldots\ldots\ldots(4)$$

substitute (2),(3) &(4) in (1)

$$f(x) = \frac{2/\pi}{2} + \frac{\cos x}{2} - \sum_{n=2}^{\infty} \frac{2\cos\frac{n\pi}{2}}{\pi(n^2-1)}\cos nx$$

$$f(x) = \frac{1}{\pi} + \frac{\cos x}{2} - \sum_{n=2}^{\infty} \frac{2\cos\frac{n\pi}{2}}{\pi(n^2-1)}\cos nx\ .$$

1.4 Complex or Exponential form of Fourier Series:

Interval	Complex Fourier Series	C_n
$(-l, l)$	$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$	$\frac{1}{2l}\int_{-l}^{l} f(x) e^{\frac{-in\pi x}{l}} dx$
$(0, 2l)$	$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$	$\frac{1}{2l}\int_{0}^{2l} f(x) e^{\frac{-in\pi x}{l}} dx$
$(-\pi, \pi)$	$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$
$(0, 2\pi)$	$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$	$\frac{1}{2\pi}\int_{0}^{2\pi} f(x) e^{-inx} dx$

Problem-13:

Find the complex form of the Fourier series f(x) = cosax in $(-\pi, \pi)$.

solution:

$(-\pi, \pi)$	$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \ldots\ldots\ldots(1)$	$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax e^{-inx} dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} \cos ax dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{(-in)^2 + a^2}\left(-in\cos ax + a\sin ax\right)\right]_{-\pi}^{\pi} \qquad \because \int e^{ax} dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$$

$$= \frac{1}{2\pi}\left[\frac{e^{-in\pi}}{-n^2+a^2}(-in\cos a\pi + a\sin a\pi) - \frac{e^{-in\pi}}{-n^2+a^2}(-in\cos a\pi - a\sin a\pi)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[-in\cos a\pi\left(e^{-in\pi}-e^{in\pi}\right) + a\sin a\pi\left(e^{-in\pi}+e^{in\pi}\right)\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[2in\cos a\pi\sin n\pi + 2a\sin a\pi\cos n\pi\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[2a\sin a\pi\cos n\pi\right] \qquad \because \sin n\pi = 0$$

$$= \frac{(-1)^n a\sin a\pi}{\pi(a^2-n^2)}$$

$$f(x) = \sum_{n=-\infty}^{\infty} e^{inx}\frac{(-1)^n a\sin a\pi}{\pi(a^2-n^2)}$$

$$f(x) = \frac{a\sin a\pi}{\pi}\sum_{n=-\infty}^{\infty} e^{inx}\frac{(-1)^n}{a^2-n^2}.$$

Problem-14:

Find the complex form of the Fourier series of e^{ax} ,where a is a constant in $(-l,l)$

solution:

given $f(x) = e^{ax}$ in $(-l,l)$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}} \text{ where } C_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{-in\pi x}{l}}dx$$

$$C_n = \frac{1}{2l}\int_{-l}^{l} e^{ax}e^{\frac{-in\pi x}{l}}dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{\left(a-\frac{-in\pi}{l}\right)x}dx$$

$$= \frac{1}{2l}\left[\frac{e^{\left(a-\frac{-in\pi}{l}\right)x}}{\left(a-\frac{-in\pi}{l}\right)}\right]_{-l}^{l}$$

$$= \frac{1}{2l}\left[\frac{e^{\left(a-\frac{in\pi}{l}\right)l}}{\left(a-\frac{in\pi}{l}\right)} - \frac{e^{\left(a-\frac{in\pi}{l}\right)-l}}{\left(a-\frac{in\pi}{l}\right)}\right]_{-l}^{l}$$

$$= \frac{1}{2l}\left[\frac{e^{(al-in\pi)}}{\left(a-\frac{in\pi}{l}\right)} - \frac{e^{(-al+in\pi)}}{\left(a-\frac{in\pi}{l}\right)}\right]$$

$$=\frac{1}{2l}\cdot\frac{l}{al-in\pi}\left[e^{al-in\pi}-e^{-al+in\pi}\right]$$

$$=\frac{1}{2(al-in\pi)}\left[e^{al-in\pi}-e^{-al+in\pi}\right]$$

$$=\frac{1}{2(al-in\pi)}\left[e^{al}e^{-in\pi}-e^{-al}e^{in\pi}\right]$$

$$=\frac{1}{2(al-in\pi)}\left[e^{al}(-1)^{n}-e^{-al}(-1)^{n}\right]$$

$$=\frac{(-1)^{n}}{2(al-in\pi)}\left[e^{al}-e^{-al}\right]$$

$$=\frac{(-1)^{n}}{al-in\pi}\left[\frac{e^{al}-e^{-al}}{2}\right]$$

$$=\frac{(-1)^{n}}{al-in\pi}\sinh al$$

$$C_n=\frac{(-1)^{n}}{al-in\pi}\sinh al$$

$$f(x)=\sum_{n=-\infty}^{\infty}\frac{(-1)^{n}}{al-in\pi}\sinh al\, e^{\frac{in\pi x}{l}}$$

$$f(x)=\sinh al\sum_{n=-\infty}^{\infty}\frac{(-1)^{n}}{al-in\pi}e^{\frac{in\pi x}{l}}$$

Problem-15:

Find the complex form of the Fourier series of $f(x)=\begin{cases}kx, 0<x<l\\ 0, l<x<2l\end{cases}$

solution:

given $f(x)=\sum_{n=-\infty}^{\infty}C_n e^{\frac{in\pi x}{l}}$ where $C_n=\frac{1}{2l}\int_0^{2l}f(x)e^{\frac{-in\pi x}{l}}dx$

$$C_n=\frac{1}{2l}\left[\int_0^{l}kxe^{\frac{-in\pi x}{l}}dx+\int_l^{2l}0.e^{\frac{-in\pi x}{l}}dx\right]$$

$$=\frac{1}{2l}\int_0^{l}kxe^{\frac{-in\pi x}{l}}dx$$

$$=\frac{k}{2l}\left[x\left(\frac{e^{\frac{-in\pi x}{l}}}{\frac{-in\pi}{l}}\right)-(1)\left(\frac{e^{\frac{-in\pi x}{l}}}{\left(\frac{-in\pi}{l}\right)^{2}}\right)\right]_0^l$$

$$= \frac{k}{2l}\left[l\left(\frac{e^{-in\pi}}{\frac{-in\pi}{l}} \right) - (1)\left(\frac{e^{-in\pi}}{\left(\frac{-in\pi}{l}\right)^2} \right) - \left(0 + \frac{l^2}{n^2\pi^2} \right) \right]_0^l$$

$$= \frac{k}{2l}\left[\frac{-l^2 e^{-in\pi}}{in\pi} + \frac{l^2 e^{-in\pi}}{n^2\pi^2} - \frac{l^2}{n^2\pi^2} \right]$$

$$= \frac{kl}{2n^2\pi^2}\left[in\pi(-1)^n + (-1)^n - 1 \right]$$

1.5PARSEVALS IDENTITY:

If the Fourier series corresponding to f(x) in $(-l, l)$,then $\frac{1}{l}\int_{-l}^{l}\{f(x)\}^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$.

PARSEVALS IDENTITY

Let f(x) be a periodic function with period 2π defined in the interval $(-\pi, \pi)$ .Then $\frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x)\}^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ where a_0, a_n & b_n are Fourier constants of f(x).

proof:

Let the Fourier series for f(x) in $(-\pi, \pi)$ be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ……….(1)

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$ ……….(1)

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$ ……….(2)

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$ ………..(3)

multiplying (1) by f(x) and integrating term by term from $-\pi$ to π ,we get

$$\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0}{2}\int_{-\pi}^{\pi} f(x)dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} f(x)\cos nx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} f(x)\sin nx$$

$= \frac{a_0}{2}[\pi a_0] + \sum_{n=1}^{\infty} a_n [\pi a_n] + \sum_{n=1}^{\infty} b_n [\pi b_n]$ using (2) ,(3)and (4)

$$= \frac{\pi a_0^2}{2} + \sum_{n=1}^{\infty}\pi a_n^2 + \sum_{n=1}^{\infty}\pi b_n^2$$

$$= \pi\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \right]$$

$$= 2\pi\left[\frac{a_0{}^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n{}^2 + b_n^2)\right]$$

$$\int_{-\pi}^{\pi}[f(x)]^2 dx = 2\pi\left[\frac{a_0{}^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n{}^2 + b_n^2)\right]$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \left[\frac{a_0{}^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n{}^2 + b_n^2)\right]$$

Problem-16:

Obtain the half range cosine series for f(x) = x in $(0,\pi)$ and hence prove that

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + ... = \frac{\pi^4}{96}.$$

solution:

The half range cosine series of f(x) in $(0,\pi)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ ……….(1)

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} xdx$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{2}\right)$$

$$= \pi$$

$a_0 = \pi$ ……….(2)

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nxdx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos o}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$\therefore a_n = \begin{cases} 0, when' n' iseven \\ \frac{-4}{n^2\pi}, when' n' isodd \end{cases} \ldots\ldots\ldots\ldots(3)$$

substitute (2),(3) in (1)

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5}^{\infty} \frac{-4}{n^2\pi}\cos nx$$

The Parsevals identity for Fourier cosine series in $(0,\pi)$ is $\frac{2}{\pi}\int_0^{\pi}\left[f(x)\right]^2 dx = \frac{a_0^2}{2} + \sum a_n^2$

here $a_0 = \pi, \therefore a_n = \frac{-4}{n^2\pi}, n \to odd$

$$\frac{2}{\pi}\int_0^{\pi}(x)^2 dx = \frac{\pi^2}{2} + \sum\left(\frac{-4}{n^2\pi}\right)^2$$

$$\frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{\pi^2}{2} + \sum\frac{16}{n^4\pi^2}$$

$$\frac{2}{\pi}\left(\frac{\pi^3}{3}\right)dx = \frac{\pi^2}{2} + \sum\frac{16}{n^4\pi^2}$$

$$\frac{2\pi^2}{3} = \frac{\pi^2}{2} + \sum\frac{16}{n^4\pi^2}$$

$$\frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots..\right)$$

$$\frac{\pi^2}{6} = \frac{16}{\pi^2}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots..\right)$$

$$\frac{\pi^4}{96} = \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots..\right)$$

$$\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots..\right) = \frac{\pi^4}{96}.$$

Problem-17:

Find the Fourier series x^2 in $(-\pi, \pi)$ .Use Parsevals identity to prove $\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots.$

Solution:

Let $f(x) = x^2$, $f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore$ f(x) is an even function in $(-\pi,\pi)$

hence $b_n = 0$

Let the required fourier series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3}\right]$$

$$= \frac{2\pi^2}{3}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx$$

$$a_n = \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) + (2x)\left(\frac{\cos nx}{n^2}\right) - 2\left(\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 + \left(\frac{2\pi(-1)^n}{n^2}\right) - 0\right) - (0 + 0 - 0)\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right]$$

$$= \frac{4}{n^2}(-1)^n$$

$$a_n = \frac{4}{n^2}(-1)^n$$

Using Parsevals identity, $\frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x)\}^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{1}{2\pi}\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{2\pi}\left[\frac{\pi^5}{5} + \frac{\pi^5}{5}\right] = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ...\right)$$

$$\frac{\pi^4}{5} - \frac{\pi^4}{9} = 8\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ...\right)$$

$$\frac{4\pi^4}{45(8)} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ...$$

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ...$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ... = \frac{\pi^4}{90}$$

1.6 HARMONIC ANALYSIS

Some times the function is not given by a formula, but by a graph or a table of corresponding values.The process of finding Euler constant for a tabular function is known as harmonic analysis. The Fourier constant are evaluated by the following formulae:

1. $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$ or $a_0 = 2\left[\frac{\sum f(x)}{n}\right]$
2. $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$ or $a_n = 2\left[\frac{\sum f(x)\cos nx}{n}\right]$
3. $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$ or $b_n = 2\left[\frac{\sum f(x)\sin nx}{n}\right]$

Fundamental or First Harmonic : The term or $(a_1 \cos x + b_1 \sin x)$ in F.S is called First Harmonic

Second Harmonic: The term or $(a_2 \cos 2x + b_2 \sin 2x)$ in F.S is called Second Harmonic and so on.

Type 1 : given data are in π form
Type 2 : given data are in degree form
Type 3 : given data are in T form
Type 4 : given data are in l form

Problems Based On Harmonic Analysis:

Type 1 : given data are in π form

Problem-18:

Find the Fourier series up to the third Harmonic for y = f(x) in $(0,2\pi)$ defined by the table of values given below.

X	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1	1.4	1.9	1.7	1.5	1.2	1.0

x	y	cos x	sin x	Cos2x	Sin 2x	Cos3x	Sin3 x	ycosx	ysinx	yCos2x	ySin2x	yCos3 x	ySin3x
0	1	1	0	1	0	1	0	1	0	1	0	1	0
$\pi/3$	1.4	0.5	0.866	-0.5	0.866	-1	0	0.7	1.2124	-0.7	1.2124	-1.4	0
$2\pi/3$	1.9	-0.5	0.866	-0.5	-0.866	1	0	-0.95	1.6454	-0.95	-1.6454	1.9	0
π	1.7	-1	0	1	0	-1	0	-1.7	0	1.7	0	-1.7	0
$4\pi/3$	1.5	-0.5	-0.866	-0.5	0.866	1	0	-0.75	-1.299	-0.75	1.299	1.5	0
$5\pi/3$	1.2	0.5	-0.866	-0.5	-0.866	-1	0	-0.6	-1.0392	-0.6	-1.0392	-1.2	0
	8.7							-1.1	0.5196	-0.3	-0.1732	0.1	0
	$\sum y$							$\sum y\cos x$	$\sum y\sin x$	$\sum y\cos 2x$	$\sum y s$	$\sum y\cos 3x$	$\sum y\sin 3x$

Since the last value of y is a repetition of the first only the first six values will be used .W.K.T the Fourier series is given by

$$y = \frac{a_0}{2} + a_1\cos x + b_1\sin x + a_2\cos 2x + b_2\sin 2x + a_3\cos 3x + b_3\sin 3x \ldots\ldots\ldots(1)$$

$$a_0 = 2\left[\frac{\sum y}{n}\right] = 2\left[\frac{8.7}{6}\right] = 2.9$$

$$a_1 = 2\left[\frac{\sum y\cos x}{n}\right] = 2\left[\frac{-1.1}{6}\right] = -0.37$$

$$a_2 = 2\left[\frac{\sum y\cos 2x}{n}\right] = 2\left[\frac{-0.3}{6}\right] = -0.1$$

$$a_3 = 2\left[\frac{\sum y\cos 3x}{n}\right] = 2\left[\frac{0.1}{6}\right] = 0.03$$

$$b_1 = 2\left[\frac{\sum y\sin x}{n}\right] = 2\left[\frac{0.5196}{6}\right] = 0.17$$

$$b_2 = 2\left[\frac{\sum y\sin 2x}{n}\right] = 2\left[\frac{-0.1732}{6}\right] = -0.06$$

$$b_3 = 2\left[\frac{\sum y \sin 3x}{n}\right] = 2\left[\frac{0}{6}\right] = 0$$

substituting these values in (1) we get

$$y = \frac{2.9}{2} + (-0.37)\cos x + (0.17)\sin x + (-0.1)\cos 2x + (-0.06)\sin 2x + (0.03)\cos 3x + (0)\sin 3x$$

$$y = 1.45 + (-0.37)\cos x + (0.17)\sin x + (-0.1)\cos 2x + (-0.06)\sin 2x + (0.03)\cos 3x$$

Type 2 : given data are in degree form

Problem-19:

Find an empirical formula of the form $f(x) = a_0 + a_1 \cos x + b_1 \sin x$ for the following data given that f(x) is periodic with period 2π .

X in degree	0	60	120	180	240	300	360
Y=f(x)	40	31	-13.7	20	3.7	-21	40

Solution:

since the last value of y is a repetition of the first only the first six values will be used

X	Y	Cosx	Sinx	Ycosx	ySinx
0	40	1	0	40	0
60	31	0.5	0.866	15.5	26.846
120	-13.7	-0.5	0.866	6.85	-11.864
180	20	-1	0	-20	0
240	3.7	-0.5	-0.866	-1.85	-3.204
300	-21	0.5	-0.866	-10.5	18.186
	$\sum y$ = 60			$\sum y$ cosx = 30	$\sum y$ sinx = 29.964

$$a_0 = 2\left[\frac{\sum y}{n}\right] = 2\left[\frac{60}{6}\right] = 20$$

$$a_1 = 2\left[\frac{\sum y \cos x}{n}\right] = 2\left[\frac{30}{6}\right] = 10$$

$$b_1 = 2\left[\frac{\sum y \sin x}{n}\right] = 2\left[\frac{29.964}{6}\right] = 9.988$$

$f(x) = 20 + 10\cos x + 9.988\sin x$.

Type 3 : given data are in T form

Formula $\theta = \dfrac{2\pi x}{T}$

Problem-20:

The values of x and the corresponding values of f(x) over a period T given below Show that

$f(x) = 0.75 + 0.37\cos\theta + 1.004\sin\theta$ where $\theta = \dfrac{2\pi x}{T}$

X	0	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{3}$	$\frac{5T}{6}$	T
Y=f(x)	1.98	1.30	1.05	1.3	-0.88	-0.25	1.98

Solution:

since the last value of y is a repetition of the first only the first six values will be used

X	θ	Y	Cos θ	Sin θ	Ycos θ	ySin θ
0	0	1.98	1	0	1.98	0
$T/6$	$\pi/3$	1.30	0.5	0.866	0.65	1.1258
$T/3$	$2\pi/3$	1.05	-0.5	0.866	-0.525	0.9093
$T/2$	π	1.30	-1	0	-1.3	0
$2T/3$	$4\pi/3$	-0.88	-0.5	-0.866	0.44	0.762
$5T/6$	$5\pi/3$	-0.25	0.5	-0.866	-0.125	0.2165
		$\sum y = 4.5$			$\sum y\cos\theta = 1.12$	$\sum y\sin\theta = 3.0136$

$$f(x) = a_0 + a_1\cos\theta + b_1\sin\theta$$

$$a_0 = 2\left[\frac{\sum y}{n}\right] = 2\left[\frac{4.5}{6}\right] = 1.5$$

$$a_1 = 2\left[\frac{\sum y\cos\theta}{n}\right] = 2\left[\frac{1.12}{6}\right] = 0.3733$$

$$b_1 = 2\left[\frac{\sum y\sin\theta}{n}\right] = 2\left[\frac{3.0136}{6}\right] = 1.0045$$

$f(x) = 0.75 + 0.373\cos\theta + 1.0045\sin\theta$, where $\theta = \frac{2\pi x}{T}$

Type 4 : given data are in l form

Problem-21:

Find the constant term and the coefficient of the first sine and cosine term in the Fourier expansion of y as given in the following table.

X	0	1	2	3	4	5
Y=f(x)	9	18	24	28	26	20

solution:

here the length of the interval is $2l = 6 \Rightarrow l = 3$

therefore the Fourier series can be represented by

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l} \right) + \left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l} \right) + \text{..........}$$

here $y = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}$ ....(1)

x	y	$\cos\frac{\pi x}{3}$	$\sin\frac{\pi x}{3}$	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$
0	9	1	0	9	0
1	18	0.5	0.866	9	15.5888
2	24	-0.5	0.866	-12	20.785
3	28	-1	0	-28	0
4	26	-0.5	-0.866	-13	-22.517
5	20	0.5	-0.866	10	-17.321
	125			-25	-3.465

$$a_0 = 2\left[\frac{\sum y}{n}\right] = 2\left[\frac{125}{6}\right] = 41.67$$

$$a_1 = 2\left[\frac{\sum y\cos\frac{\pi x}{3}}{n}\right] = 2\left[\frac{-25}{6}\right] = -8.33$$

$$b_1 = 2\left[\frac{\sum y\sin\frac{\pi x}{3}}{n}\right] = 2\left[\frac{-3.465}{6}\right] = -1.16$$

substituting these values in (1) we get

$$y = \frac{41.67}{2} - 8.33\cos\frac{\pi x}{3} - 1.16\sin\frac{\pi x}{3}$$

$$y = 20.84 - 8.33\cos\frac{\pi x}{3} - 1.16\sin\frac{\pi x}{3}.$$

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UNIT III - FOURIER TRANSFORM

1) Write Fourier Transform pair.

The Fourier Transform of $f(x)$ is defined as $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = F(s)$.

The Inverse Fourier Transform of $F(s)$ is defined as $F^{-1}[F(s)] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} ds$

2) Find the fourier transform of $f(x-a)$.

Solution: $F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a) e^{isx}\, dx$

$$x - a = t \Rightarrow x = a + t \Rightarrow dx = dt$$

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(t) e^{i(a+t)s}\, dt$$

$$= e^{ias} \int_{-\infty}^{\infty} f(t) e^{ist}\, dt$$

$$F[f(x-a)] = e^{ias} F(s)$$

3) Find the fourier transform of $f(x)\cos ax$ or State and prove modulation theorem

Solution: $F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax\, e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left(\frac{e^{iax} + e^{-iax}}{2} \right) e^{isx} dx$$

$$= \frac{1}{2} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{iax}\, e^{isx} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iax}\, e^{isx} dx \right)$$

$$= \frac{1}{2} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s-a)x} dx \right)$$

$$= \frac{1}{2} [F(s+a) + F(s-a)]$$

4) Define convolution

The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x)*g(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt\,.$$

1) Find the Fourier transform of $f(x)=\begin{cases}1,|x|\le 1\\ 0,|x|>1\end{cases}$**. Hence prove that** $\int_0^{\infty}\frac{\sin t}{t}\,dt=\frac{\pi}{2}=\int_0^{\infty}\frac{\sin^2 t}{t^2}\,dt$

Solution: Given $f(x)=\begin{cases}1,|x|\le 1\\ 0,|x|>1\end{cases}$

$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\cos sx\,dx \qquad \because \int_{-1}^{1}\sin sx\,dx=0 \text{ because } \sin sx \text{ is odd.}$$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{1}\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{\sin sx}{s}\right]_0^1$$

$$F(s)=\sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s}\right]$$

Taking inverse fourier transform, $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds=f(x)$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s}\right]e^{-isx}ds=1$$

Put $x=0$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin s}{s}\,ds=1$$

$$\frac{2}{\pi}\int_0^{\infty}\frac{\sin s}{s}\,ds = 1$$

$$\int_0^{\infty}\frac{\sin s}{s}\,ds = \frac{\pi}{2}$$

Put $s = t$

$$\int_0^{\infty}\frac{\sin t}{t}\,dt = \frac{\pi}{2}$$

Using Parseval's Identity, $\int_{-\infty}^{\infty}|F(s)|^2\,ds = \int_{-\infty}^{\infty}|f(x)|^2\,dx$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 s}{s^2}\,ds = \int_{-1}^{1}dx$$

$$\frac{4}{\pi}\int_0^{\infty}\frac{\sin^2 s}{s^2}\,ds = 2\int_0^{1}dx$$

$$= 2[x]_0^1$$

$$= 2$$

$$\frac{2}{\pi}\int_0^{\infty}\frac{\sin^2 s}{s^2}\,ds = 1$$

$$\int_0^{\infty}\frac{\sin^2 s}{s^2}\,ds = \frac{\pi}{2}$$

Put $s = t$

$$\int_0^{\infty}\frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$$

2) Show that the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$ **is** $2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)$**.**

Hence deduce that $\int_0^{\infty}\frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4}$**. Also find the value of** $\int_0^{\infty}\frac{(\sin t - t\cos t)^2}{t^6}\,dt$ **.**

Solution: Given $f(x)=\begin{cases} a^2-x^2, & |x|<a \\ 0 & ,|x|>a>0 \end{cases}$

$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\left(a^2-x^2\right)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\left(a^2-x^2\right)\cos sx\,dx \qquad \because \int_{-a}^{a}\left(a^2-x^2\right)\sin sx\,dx=0 \text{ because } \sin sx \text{ is odd.}$$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{a}\left(a^2-x^2\right)\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\left(a^2-x^2\right)\left(\frac{\sin sx}{s}\right)-(-2x)\left(\frac{-\cos sx}{s^2}\right)+(-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[-\frac{2a\cos as}{s^2}+\frac{2\sin as}{s^3}\right]$$

$$F(s)=2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as-as\cos as}{s^3}\right)$$

Taking inverse fourier transform, $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds=f(x)$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as-as\cos as}{s^3}\right)e^{-isx}ds=a^2-x^2$$

Put $x=0$ and $a=1$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s-s\cos as}{s^3}\right)ds=1$$

$$\frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin s-s\cos s}{s^3}\right)ds=1$$

$$\int_{0}^{\infty}\left(\frac{\sin s-s\cos s}{s^3}\right)ds=\frac{\pi}{4}$$

Put $s=t$

$$\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4}$$

Using Parseval's Identity, $\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$

$$\frac{8}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds = \int_{-a}^{a} \left(a^2 - x^2\right)^2 dx$$

$$\frac{16}{\pi}\int_0^\infty \frac{(\sin as - as\cos as)^2}{s^6}\,ds = 2\int_0^a (a^2 - x^2)^2\,dx$$

$$= 2\int_0^a (a^4 - 2a^2x^2 + x^4)dx$$

$$= 2\left[(a^4 x - 2a^2\frac{x^3}{3} + \frac{x^5}{5})\right]_0^a$$

$$= 2\left(a^5 - \frac{2}{3}a^5 + \frac{a^5}{5}\right)$$

$$= \frac{16a^5}{15}$$

Put $a = 1$

$$\frac{16}{\pi}\int_0^\infty \frac{(\sin s - s\cos s)^2}{s^6}\,ds = \frac{16}{15}$$

$$\int_0^\infty \frac{(\sin s - s\cos s)^2}{s^6}\,ds = \frac{\pi}{15}$$

Put $s = t$

$$\int_0^\infty \frac{(\sin t - t\cos t)^2}{t^6}\,ds = \frac{\pi}{15}$$

3) Find the Fourier transform of $f(x) = \begin{cases} a - |x|, & |x| \le a \\ 0, & |x| > a \end{cases}$. Hence find the value of $\int_0^\infty \frac{\sin^2 x}{x^2}dx$

$\int_0^\infty \frac{\sin^4 x}{x^4}dx$.

Solution: Given $f(x)=\begin{cases} a-|x|, |x|\le a \\ 0 \quad ,|x|>a \end{cases}$

$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\left(a-|x|\right)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\left(a-|x|\right)\cos sx\,dx \qquad \because \int_{-a}^{a}\left(a-|x|\right)\sin sx\,dx = 0 \text{ because } \sin sx \text{ is odd.}$$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{a}\left(a-x\right)\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[(a-x)\left(\frac{\sin sx}{s}\right)-(-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[-\frac{\cos as}{s^2}+\frac{1}{s^2}\right]$$

$$F(s)=2\sqrt{\frac{2}{\pi}}\left[\frac{\sin^2\frac{as}{2}}{s^2}\right]$$

Taking inverse fourier transform, $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds = f(x)$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}2\sqrt{\frac{2}{\pi}}\left[\frac{\sin^2\frac{as}{2}}{s^2}\right]e^{-isx}ds = a-|x|$$

Put $x=0$ and $a=1$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2\frac{s}{2}}{s^2}ds=1$$

$$\frac{4}{\pi}\int_{0}^{\infty}\frac{\sin^2\frac{s}{2}}{s^2}ds=1$$

$$\int_0^\infty \frac{\sin^2 \frac{s}{2}}{s^2}\, ds = \frac{\pi}{4}$$

Put $\frac{s}{2} = x \Rightarrow s = 2x \Rightarrow ds = 2dx$

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$$

Using Parseval's Identity, $\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$

$$\frac{8}{\pi} \int_{-\infty}^{\infty} \left[\frac{\sin^4 \frac{as}{2}}{s^4} \right] ds = \int_{-a}^{a} \left(a - |x|\right) dx$$

$$\frac{16}{\pi} \int_0^\infty \frac{\sin^4 \frac{as}{2}}{s^4}\, ds = 2\int_0^a (a-x)^2\, dx$$

$$= 2\int_0^a (a^2 - 2ax + x^2) dx$$

$$= 2\left[(a^2 x - ax^2 + \frac{x^3}{3}) \right]_0^a$$

$$= \frac{2}{3} a^3$$

Put $a = 1$

$$\frac{16}{\pi} \int_0^\infty \frac{\sin^4 \frac{s}{2}}{s^4}\, ds = \frac{2}{3}$$

Put $\frac{s}{2} = x \Rightarrow s = 2x \Rightarrow ds = 2dx$

$$\int_0^\infty \frac{\sin^4 x}{x^4}\, dx = \frac{\pi}{3}$$

4) Find fourier transform of $e^{-a^2x^2}$. Hence prove that $e^{-\frac{x^2}{2}}$ is self reciprocal.

Solution: $f(x) = e^{-a^2x^2}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} dx$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

Put $ax - \frac{is}{2a} = t \quad \Rightarrow \quad adx = dt \quad \Rightarrow dx = \frac{dt}{a}$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} \sqrt{\pi}$$

$$F[e^{-a^2x^2}] = \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2}}$$

Put $a = \frac{1}{\sqrt{2}}$

$$F[e^{-\frac{x^2}{2}}] = e^{-\frac{s^2}{2}}$$

5) Find the Fourier transform of $e^{-a|x|}$, $a > 0$. Hence deduce that $F[xe^{-a|x|}] = i\sqrt{\frac{2}{\pi}} \frac{2as}{(a^2+s^2)^2}$.

Solution: Given $f(x) = e^{-a|x|}$ if $a > 0$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{isx} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-ax} \cos sx \; dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2 + s^2} (-a \cos sx + s \sin sx) \right]_0^{\infty}$$

$$F(s) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2}$$

$$F[xf(x)] = -i \frac{d}{ds} F(s)$$

$$= -i \frac{d}{ds} \left(\sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2} \right)$$

$$= -i \sqrt{\frac{2}{\pi}} a \frac{d}{ds} (a^2 + s^2)^{-1}$$

$$= -i \sqrt{\frac{2}{\pi}} a(-1)(a^2 + s^2)^{-2} 2s$$

$$= i \sqrt{\frac{2}{\pi}} \frac{2as}{(a^2 + s^2)^2}$$

SINE AND COSINE TRANSFORMS

Write fourier cosine transform pair.

The fourier cosine transform of $f(x)$ is defined as $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx \, dx = F_c(s)$.

The Inverse fourier cosine transform of $F_c(s)$ is defined as

$$F_c^{-1}[F_c(s)] = f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(s) \cos sx \, ds .$$

Write fourier sine transform pair.

The fourier sine transform of $f(x)$ is defined as $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx = F_s(s)$.

The Inverse fourier sine transform of $F_c(s)$ is defined as

$F_s^{-1}[F_s(s)] = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$.

Parseval's Identity for single function

Cosine transform for single function: $\int_0^\infty |f(x)|^2 \, dx = \int_0^\infty |F_c(s)|^2 \, ds$

Sine transform for single function: $\int_0^\infty |f(x)|^2 \, dx = \int_0^\infty |F_s(s)|^2 \, ds$

Parseval's Identity for two functions

If $F_c(s)$ **and** $G_c(s)$ **are fourier cosine transforms of** $f(x)$ and $g(x)$ respectively then

$\int_0^\infty f(x) g(x) \, dx = \int_0^\infty F_c(s) G_c(s) \, ds$.

If $F_s(s)$ **and** $G_s(s)$ **are fourier cosine transforms of** $f(x)$ and $g(x)$ respectively then

$\int_0^\infty f(x) g(x) \, dx = \int_0^\infty F_s(s) G_s(s) \, ds$

1. **Find fourier sine and cosine transform of** e^{-ax} **and hence deduce the inversion formula. Also evaluate** $\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx$ **and** $\int_0^\infty \frac{1}{(x^2+a^2)^2} dx$ **.**

Solution: $f(x) = e^{-ax}$ $a > 0$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{s^2 + a^2} (-a \sin sx - s \cos sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2 + a^2}$$

Taking inverse fourier transform, $\sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds = f(x)$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2 + a^2} \sin sx \, ds = e^{-ax}$$

$$\frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{s^2 + a^2} ds = e^{-ax}$$

Put $s = x$ and $x = \alpha$

$$\int_0^\infty \frac{x \sin \alpha x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-\alpha x}$$

Using Parseval's Identity for sine transform, $\int_0^\infty |F_s(s)|^2 \, ds = \int_0^\infty | f(x) |^2 \, dx$

$$\frac{2}{\pi} \int_0^\infty \frac{s^2}{(a^2 + s^2)^2} \, ds = \int_0^\infty e^{-2ax} \, dx$$

$$= \left[\frac{e^{-2ax}}{-2a} \right]_0^\infty$$

$$= \frac{1}{2a}$$

$$\int_0^\infty \frac{s^2}{(a^2 + s^2)^2} \, ds = \frac{\pi}{4a}$$

Put $s = x$

$$\int_0^\infty \frac{x^2}{(a^2 + x^2)^2} \, dx = \frac{\pi}{4a}.$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{s^2 + a^2} (-a \cos sx + s \sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2 + a^2}$$

Taking inverse fourier transform, $\sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds = f(x)$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{s^2 + a^2} \cos sx \, ds = e^{-ax}$$

$$\frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{s^2 + a^2} ds = e^{-ax}$$

$$\int_0^\infty \frac{\cos sx}{s^2 + a^2} ds = \frac{\pi}{2a} e^{-ax}$$

Put $s = x$ and $x = \alpha$

$$\int_0^\infty \frac{\cos \alpha x}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-a\alpha}$$

Using Parseval's Identity for cosine transform, $\int_0^\infty |F_c(s)|^2 \, ds = \int_0^\infty | f(x) |^2 \, dx$

$$\frac{2}{\pi} \int_0^\infty \frac{a^2}{(a^2 + s^2)^2} \, ds = \int_0^\infty e^{-2ax} \, dx$$

$$= \left[\frac{e^{-2ax}}{-2a} \right]_0^\infty$$

$$= \frac{1}{2a}$$

$$\int_0^\infty \frac{a^2}{(a^2 + s^2)^2} \, ds = \frac{\pi}{4a}$$

Put $s = x$

$$\int_0^\infty \frac{a^2}{(a^2 + x^2)^2} \, dx = \frac{\pi}{4a^3} .$$

2. Evaluate $\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)} dx$.

Solution: Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{s^2+a^2} (-a\cos sx + s\sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

Similarly $F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}} \frac{b}{s^2+b^2}$

Using Parseval's Identity, $\int_0^\infty F_c(s) G_c(s) \, ds = \int_0^\infty f(x) \, g(x) \, dx$

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \sqrt{\frac{2}{\pi}} \frac{b}{s^2+b^2} ds = \int_0^\infty e^{-ax} e^{-bx} \, dx$$

$$\frac{2ab}{\pi} \int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)} ds = \int_0^\infty e^{-(a+b)x} \, dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^\infty$$

$$= \frac{1}{a+b}$$

$$\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)} ds = \frac{\pi}{2ab(a+b)}$$

Put $s = x$

$$\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{2ab(a+b)} .$$

3. Find the fourier sine transform of $\frac{x}{x^2+a^2}$ and fourier cosine transform of $\frac{a}{x^2+a^2}$.

Solutiion: Let $f(x) = e^{-ax}$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{s^2+a^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$$

Taking inverse cosine fourier transform, $\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\,ds = f(x)$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\cos sx\,ds = e^{-ax}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{a}{s^2+a^2}\cos sx\,ds = \sqrt{\frac{\pi}{2}}e^{-ax}$$

Put $s = x$ and $x = s$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{a}{x^2+a^2}\cos sx\,dx = \sqrt{\frac{\pi}{2}}e^{-as}$$

$$F_c\left(\frac{a}{x^2+a^2}\right) = \sqrt{\frac{\pi}{2}}e^{-as}$$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{s^2+a^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$

$$=\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$$

Taking inverse sine fourier transform, $\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,ds = f(x)$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\sin sx\,ds = e^{-ax}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{s}{s^2+a^2}\sin sx\,ds = \sqrt{\frac{\pi}{2}}e^{-ax}$$

Put $s = x$ and $x = s$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x}{x^2+a^2}\sin sx\,dx = \sqrt{\frac{\pi}{2}}e^{-ax}$$

$$F_s\left(\frac{x}{x^2+a^2}\right) = \sqrt{\frac{\pi}{2}}e^{-as}.$$

4. Find the Fourier cosine transform of $f(x)=\begin{cases}1-x^2, & 0<x<1\\ 0 & ,elsewhere\end{cases}$. Hence deduce the value of $\int_0^\infty \frac{\sin x - x\cos x}{x^3}dx \quad \int_0^\infty \frac{(x\cos x-\sin x)^2}{x^6}dx$.

Soution: Given $f(x)=\begin{cases}1-x^2, & 0<x<1\\ 0 & ,elsewhere\end{cases}$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^1 \left(1-x^2\right)\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\left(1-x^2\right)\left(\frac{\sin sx}{s}\right)-(-2x)\left(\frac{-\cos sx}{s^2}\right)+(-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^1$$

$$=\sqrt{\frac{2}{\pi}}\left[-\frac{2\cos s}{s^2}+\frac{2\sin s}{s^3}\right]$$

$$F(s)=2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)$$

Taking inverse fourier transform, $\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\,ds = f(x)$

$$\sqrt{\frac{2}{\pi}}\int_0^1 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,ds = 1 - x^2$$

Put $x = 0$

$$\frac{4}{\pi}\int_0^1\left(\frac{\sin s - s\cos s}{s^3}\right)ds = 1$$

$$\int_0^1\left(\frac{\sin s - s\cos s}{s^3}\right)ds = \frac{\pi}{4}$$

Put $s = x$

$$\int_0^1 \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4}$$

Using Parseval's Identity, $\int_{-\infty}^{\infty}|F(s)|^2\,ds = \int_{-\infty}^{\infty}|f(x)|^2\,dx$

$$\frac{8}{\pi}\int_0^1\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \int_0^1 \left(1 - x^2\right)^2 dx$$

$$= \int_0^1 (1 - 2x^2 + x^4)dx$$

$$= \left[x - 2\frac{x^3}{3} + \frac{x^5}{5}\right]_0^1$$

$$= \left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= \frac{8}{15}$$

$$\int_0^1 \frac{(\sin s - s\cos s)^2}{s^6}\,ds = \frac{\pi}{15}$$

Put $s = x$

$$\int_0^1 \frac{(\sin x - x\cos x)^2}{x^6}\, dx = \frac{\pi}{15}$$

5. Find fourier cosine transform of $f(x) = e^{-a^2x^2}$ and hence fourier cosine transform of $e^{-\frac{x^2}{2}}$ and fourier sine transform of $xe^{-\frac{x^2}{2}}$.

Solution: $f(x) = e^{-a^2x^2}$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx\, dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} \cos sx\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} R.P(e^{isx})dx$$

$$= R.P \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} dx$$

$$= R.P \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

Put $ax - \frac{is}{2a} = t \quad \Rightarrow\ adx = dt \quad \Rightarrow dx = \frac{dt}{a}$

$$= R.P \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$= R.P \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= R.P \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} \sqrt{\pi}$$

$$F_c[e^{-a^2x^2}] = \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2}}$$

Put $a = \frac{1}{\sqrt{2}}$

$$F_c[e^{-\frac{x^2}{2}}] = e^{-\frac{s^2}{2}}$$

Hence $e^{-\frac{x^2}{2}}$ is self reciprocal under cosine transform.

$$F_s[xf(x)] = -\frac{d}{ds}F_c[f(x)]$$

$$F_s[xe^{-\frac{x^2}{2}}] = -\frac{d}{ds}F_c[e^{-\frac{x^2}{2}}]$$

$$= -\frac{d}{ds}e^{-\frac{s^2}{2}}$$

$$= -e^{-\frac{s^2}{2}}(-s)$$

$$F_s[xe^{-\frac{x^2}{2}}] = se^{-\frac{s^2}{2}}$$

Hence $xe^{-\frac{x^2}{2}}$ is self reciprocal under sine transform.

PROPERTIES

1. Find the Fourier sine transform of $f(x)\sin ax$

Proof: $F_s[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ax\, sinsx\, dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\left[\cos(s-a)x - \cos(s+a)x\right]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)x\,dx\right]$$

$$F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_s(s+a)]$$

2. Find the Fourier cosine transform of $f(x)\sin ax$ **.**

Proof: $F_c[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ax\cos sx\, dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\left[\,\sin(s+a)x - \sin(s-a)x\right]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s+a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s-a)x\,dx\right]$$

$$F_c[f(x)\sin ax] = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$$

3. Find the Fourier sine transform of $f(ax)$

Proof: $F_s[f(ax)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(ax)\sin sx\,dx$

$$ax = t \;\Rightarrow\; x = \frac{t}{a} \;\Rightarrow\; dx = \frac{dt}{a}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin s\frac{t}{a}\,\frac{dt}{a}$$

$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\frac{s}{a}t\;dt$$

$$F_s[f(ax)] = \frac{1}{a}F_s\left(\frac{s}{a}\right)$$

UNIT IV APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

(BOUNDARY VALUE PROBLEMS)

1. Classify the following partial differential equations

$$(a)\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$$

$$(b)\ \frac{\partial^2 u}{\partial x\partial y}=\left(\frac{\partial u}{\partial x}\right)+\left(\frac{\partial u}{\partial y}\right)+xy$$

Solution:

$$(a)A=1, B=0, C=-1$$
$$B^2-4AC=0+4=4>0$$

Equation is hyperbolic

$$(b)A=0, B=1, C=0$$
$$B^2-4AC=1-0>0$$

Equation is hyperbolic

2. Classify the following second order partial differential equations

$$(a)4\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}-6\frac{\partial u}{\partial x}-8\frac{\partial u}{\partial y}-16u=0$$

$$(b)\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$$

Solution:

$$(a)A=4, B=4, C=1$$
$$B^2-4AC=16-16=0$$

Equation is Parabolic

$$(b)A=1, B=0, C=1$$
$$B^2-4AC=-4<0$$

Equation is Elliptic

VIBRATION OF STRING

1) What is the constant a^2 in wave equation $u_{tt} = a^2 u_{xx}$?

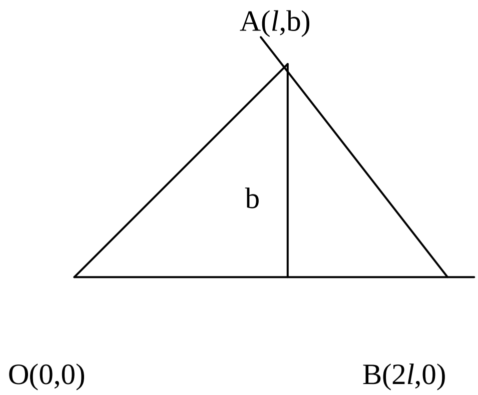
Solution: $a^2 = \frac{T}{m} = \frac{Tension}{\text{Mass per unit length of the string}}$

2) A tightly stretched string of length $2l$ is fastened at both ends. The midpoint of the string is displaced to a distance b and released from rest in this position. Write the initial conditions.

Solution:

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The equation of OA is



$$\frac{y-b}{-b} = \frac{x-l}{l}$$
$$\Rightarrow y - b = \frac{lb - bx}{l}$$
$$\Rightarrow y = \frac{lb - bx + lb}{l} = \frac{b}{l}(2l - x), 0 < x < l$$

The equation of AB is

$$\frac{y-0}{b} = \frac{x-0}{l-0}$$
$$\Rightarrow y = \frac{bx}{l}, l < x < 2l$$

The initial boundary conditions are

$$(i)\ y(0,t) = 0$$
$$(ii)\ y(l,t) = 0$$
$$(iii)\ \frac{\partial y(x,0)}{\partial t} = 0$$

$$(iv)\ y(x,0) = \begin{cases} \frac{bx}{l}, 0 < x < l \\ \frac{2}{l}(2l - x), l < x < 2l \end{cases}$$

Problems on vibrating on strings with initial velocity zero

1) A string is stretched and fastened to two points $x=0$ and $x=l$ apart motion is started by displacing the string into the form $y=k(lx-x^2)$ from which it is released at time $t=0$. Find the displacement of any point on the string at a distance of x from one end at time t.

Solution:

The wave equation is $\dfrac{\partial^2 y}{\partial t^2}=a^2\dfrac{\partial^2 y}{\partial x^2}$

From the given problem ,we get the following boundary conditions.

$$(i)\, y(0,t)=0 \,\forall t>0$$
$$(ii)\, y(l,t)=0 \,\forall t>0$$
$$(iii)\, \frac{\partial y(x,0)}{\partial t}=0 \qquad (\because \text{ initial velocity is zero})$$
$$(iv)\, y(x,0)=k(lx-x^2)$$

The correct solution which satisfies our boundary conditions is given by

$$y(x,t)=(c_1\cos px+c_2\sin px)(c_3\cos pat+c_4\sin pat)\rightarrow(1)$$

Applying (i) in (1) we get

$$y(x,t)=c_1(c_3\cos pat+c_4\sin pat)=0$$
$$(i.e)\, c_1=0 \;\&\; (c_3\cos pat+c_4\sin pat)\neq 0$$

putting $c_1=0$ in (1) we get

$$y(x,t)=c_2\sin px(c_3\cos pat+c_4\sin pat)\rightarrow(2)$$

Applying (ii) in (2) we get

$$y(l,t)=c_2\sin pl(c_3\cos pat+c_4\sin pat)$$
$$c_3\cos pat+c_4\sin pat\neq 0,$$

either $c_2=0$ or $\sin pl=0$

Suppose if we take $c_2=0$ and already $c_1=0$ then we get a trivial solution.

$$\therefore \sin pl=0$$
$$pl=n\pi$$
$$p=\frac{n\pi}{l}$$

Now substituting $p = \frac{n\pi}{l} \rightarrow (2)$ we get

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi at}{l} + c_4 \sin\frac{n\pi at}{l}\right) \rightarrow (3)$$

Before applying condition (iii) differentiate (3) partially w.r.to t

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left(-c_3 \sin\frac{n\pi at}{l}\cdot\frac{n\pi a}{l} + c_4 \cos\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right)$$

Applying (iii) we get

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left(c_4 \frac{n\pi a}{l}\right) = 0$$

$c_2 \neq 0$ (already explained)

$\sin\frac{n\pi x}{l} \neq 0$

$\frac{n\pi a}{l} \neq 0$ (all are constants)

$\therefore c_4 = 0$

$$(3) \Rightarrow y(x,t) = c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$y(x,t) = c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (4), where c_n = c_2 c_3$$

The general solution of (4) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (5)$$

Applying (iv) in (5) we get $y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = k(lx - x^2) \rightarrow (6)$

To find c_n expand in a half range fourier sine series in an interval $(0,l)$

$$f(x) = k(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \rightarrow (7)$$

$$where b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

From (6) & (7) we get $b_n = c_n$

$$c_n = \frac{2}{l}\int_0^l k\left(lx - x^2\right)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left[\left(lx - x^2\right) - \cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi} + (l-2x)\sin\frac{n\pi x}{l}\cdot\frac{l^2}{n^2\pi^2} + 2\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\cos\frac{n\pi x}{l}\cdot\frac{l^3}{n^3\pi^3}\right]_0^l$$

$$= \frac{2k}{l}\left[-2(-1)^n\frac{l^3}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3}\right] = \frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right] = \frac{8kl^2}{n^3\pi^3}\,if\ n\,is\,odd$$

Substitute c_n in (5) we get

$$y(x,t) = \sum_{n=1,3,5...}^{\infty}\frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

2. A string is stretched and its ends are fastened at two points $x=0$ and $x=l$ .The midpoint of the string is displaced transversely through a small distance b and string is released from the rest in that position. Find an expression for the transverse displacement of the string at anytime during the subsequent motion.

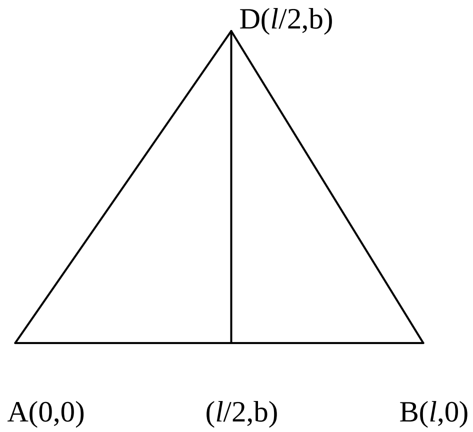
Solution: To find the equation of the string in its initial position.

The equation of the string AD is

$$\frac{x-0}{0-\frac{l}{2}} = \frac{y-0}{b-0}$$

$$\frac{2x}{l} = \frac{y}{b}$$

$$y = \frac{2xb}{l}, 0 < x < \frac{l}{2}$$



The equation of the string DB is`

$$\frac{x-\frac{l}{2}}{l-\frac{l}{2}}=\frac{y-b}{0-b}$$
$$\frac{2x-l}{l}=\frac{y-b}{-b}$$
$$y=\frac{2b}{l}(l-x),\frac{l}{2}<x<l$$

Hence initially the displacement of the string is in the form $y(x,0)=\begin{cases}\frac{2bx}{l},0<x<\frac{l}{2}\\ \frac{2b}{l}(l-x),\frac{l}{2}<x<l\end{cases}$

The wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem ,we get the following boundary conditions.

$$(i)y(0,t)=0\forall t>0$$
$$(ii)y(l,t)=0\forall t>0$$
$$(iii)\frac{\partial y(x,0)}{\partial t}=0$$
$$(iv)y(x,0)=\begin{cases}\frac{2bx}{l},0<x<\frac{l}{2}\\ \frac{2b}{l}(l-x),\frac{l}{2}<x<l\end{cases}$$

The correct solution which satisfies our boundary conditions is given by

$$y(x,t)=(c_1\cos px+c_2\sin px)(c_3\cos pat+c_4\sin pat)\to(1)$$

Applying (i) in (1) we get

$$y(x,t)=c_1(c_3\cos pat+c_4\sin pat)=0$$
$$(i.e)c_1=0\,\&\,(c_3\cos pat+c_4\sin pat)\neq 0$$

putting $c_1=0$ in (1) we get

$$y(x,t)=c_2\sin px(c_3\cos pat+c_4\sin pat)\to(2)$$

Applying (ii) in (2) we get

$$y(l,t)=c_2\sin pl(c_3\cos pat+c_4\sin pat)$$
$$c_3\cos pat+c_4\sin pat\neq 0,$$

either $c_2 = 0$ or $\sin pl = 0$

Suppose if we take $c_2 = 0$ and already $c_1 = 0$ then we get a trivial solution.

$$\therefore \sin pl = 0$$
$$pl = n\pi$$
$$p = \frac{n\pi}{l}$$

Now substituting $p = \frac{n\pi}{l} \rightarrow (2)$ we get

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi at}{l} + c_4 \sin\frac{n\pi at}{l}\right) \rightarrow (3)$$

Before applying condition (iii) differentiate (3) partially w.r.to t

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left(-c_3 \sin\frac{n\pi at}{l}\cdot\frac{n\pi a}{l} + c_4 \cos\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right)$$

Applying (iii) we get

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left(c_4 \frac{n\pi a}{l}\right) = 0$$

$c_2 \neq 0$ (already explained)

$$\sin\frac{n\pi x}{l} \neq 0$$

$\frac{n\pi a}{l} \neq 0$ (all are constants)

$$\therefore c_4 = 0$$

$$(3) \Rightarrow y(x,t) = c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$
$$y(x,t) = c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (4), where c_n = c_2 c_3$$

The general solution of (4) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (5)$$

Applying (iv) in (5) we get

$$y(x,t)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = f(x) \rightarrow (7)$$

$$Where f(x)=\begin{cases} \frac{2bx}{l}, 0<x<\frac{l}{2} \\ \frac{2b}{l}(l-x), \frac{l}{2}<x<l \end{cases}$$

To find c_n expand $f(x)$ in a half range fourier sine series in a interval $(0,l)$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \rightarrow (7)$$

$$where b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

From (6)&(7) we get $b_n = c_n$

$$c_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} \frac{2bx}{l}\sin\frac{n\pi x}{l}dx + \int_{\frac{l}{2}}^{l} \frac{2b(l-x)}{l}\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{4b}{l^2}\left[\left(x \times -\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi} + \sin\frac{n\pi x}{l}\cdot\frac{l^2}{n^2\pi^2}\right)_0^{\frac{l}{2}} + \left((l-x)-\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi} - \sin\frac{n\pi x}{l}\cdot\frac{l^2}{n^2\pi^2}\right)_{\frac{l}{2}}^{l}\right]$$

$$= \frac{4b}{l^2}\left\{\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$c_n = \frac{8b}{l^2}\cdot\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$(5) \Rightarrow y(x,t) = \sum_{n=1}^{\infty}\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

Problems on vibrating on strings with non zero initial velocity

1.A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest initially at rest in its equilibrium position.If it is set vibrating by giving each point a velocity $kx(l-x)$.Find the displacement of the string at any time.

Solution: The wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem ,we get the following boundary conditions.

$$(i)y(0,t)=0\forall t>0$$
$$(ii)y(l,t)=0\forall t>0$$
$$(iii)y(x,0)=0$$
$$(iv)\frac{\partial y(x,0)}{\partial t}=kx(l-x)$$

The correct solution which satisfies our boundary conditions is given by

$$y(x,t)=(c_1\cos px+c_2\sin px)(c_3\cos pat+c_4\sin pat)\rightarrow(1)$$

Applying (i) in (1) we get

$$y(x,t)=c_1(c_3\cos pat+c_4\sin pat)=0$$
$$(i.e)c_1=0\ \&\ (c_3\cos pat+c_4\sin pat)\neq 0$$

putting $c_1=0$ in (1) we get

$$y(x,t)=c_2\sin px(c_3\cos pat+c_4\sin pat)\rightarrow(2)$$

Applying (ii) in (2) we get

$$y(l,t)=c_2\sin pl(c_3\cos pat+c_4\sin pat)$$
$$c_3\cos pat+c_4\sin pat\neq 0,$$

either $c_2=0$ or $\sin pl=0$

Suppose if we take $c_2=0$ and already $c_1=0$ then we get a trivial solution.

$$\therefore\ \sin\ pl\ =\ 0$$
$$pl\ =\ n\pi$$
$$p\ =\ \frac{n\pi}{l}$$

Now substituting $p=\frac{n\pi}{l}\rightarrow(2)$ we get

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi at}{l} + c_4 \sin\frac{n\pi at}{l}\right) \to (3)$$

Applying (iii) in (3)we get

$$y(x,0) == c_2 \sin\frac{n\pi x}{l} c_3 = 0$$

$$\Rightarrow c_3 = 0$$

Substitute this value in (3) we get

$$y(x,0) = c_2 c_4 \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$y(x,0) = c_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \to (4) where c_n = c_2 c_4$$

The general soln of (4) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \to (5)$$

Diff partially (5) w.r.to t we get

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}$$

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l}\sin\frac{n\pi x}{l} = kx(l-x) \to (6)$$

To find c_n, Expand $kx(l-x)$ in a Fourier sine series in $(0,l)$

$$f(x) = kx(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \to (7)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

Comparing (6)&(7) we get

$$c_n \frac{n\pi a}{l} = b_n$$

$$b_n = \frac{2}{l}\int_0^l kx(l-x)\sin\frac{n\pi x}{l}dx$$

$$b_n = \frac{2k}{l}\left[-\left(lx - x^2\right)\cos\frac{n\pi x}{l}\frac{l}{n\pi} + (l-2x)\sin\frac{n\pi x}{l}\frac{l^2}{n^2\pi^2} - 2\cos\frac{n\pi x}{l}\frac{l^3}{n^3\pi^3}\right]_0^l$$

$$b_n = \frac{2k}{l}\left[(-2)(-1)^n \frac{l^3}{n^3\pi^3} + 2\frac{l^3}{n^3\pi^3}\right] = \frac{4kl^2}{n^3\pi^3}\left\{1-(-1)^n\right\}$$

$$c_n \frac{n\pi a}{l} = \frac{4kl^2}{n^3\pi^3}\left\{1-(-1)^n\right\}$$

$$c_n = \frac{4kl^3}{n^4\pi^4 a}\left\{1-(-1)^n\right\}$$

$$c_n = \begin{cases} 0 n is even \\ \dfrac{8kl^3}{n^4\pi^4 a} \quad n is odd \end{cases}$$

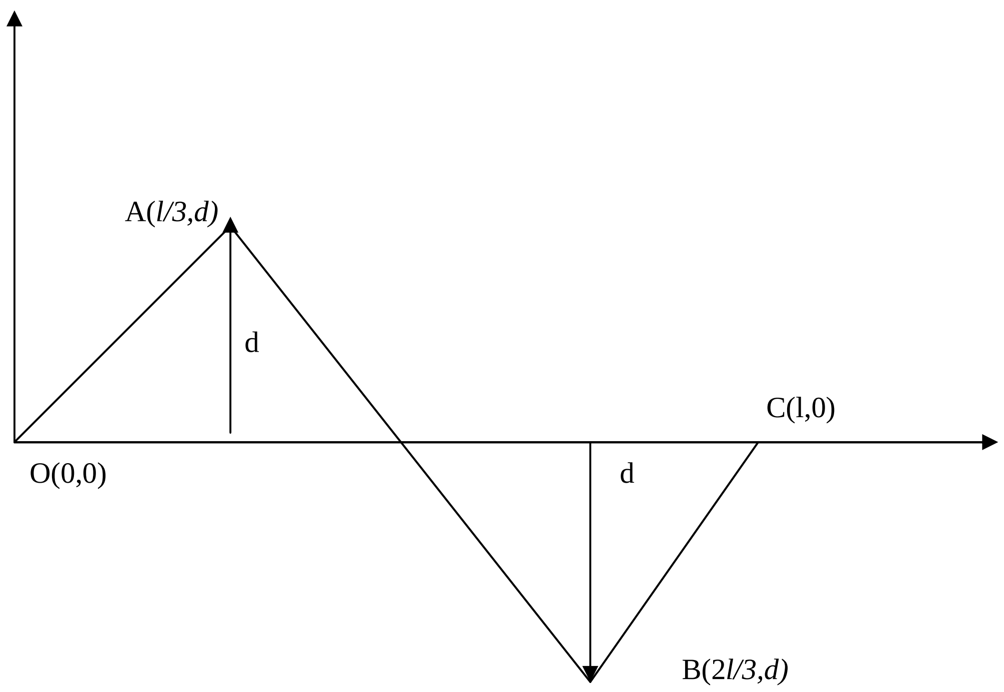
$$y(x,t) = \sum_{n=odd} \frac{8kl^3}{n^4\pi^4 a}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$y(x,t) = \frac{8kl^3}{\pi^4 a}\sum_{n=odd}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$y(x,t) = \frac{8kl^3}{\pi^4 a}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{l}\sin\frac{(2n-1)\pi at}{l}$$

2) The points of trisection of a tightly stretched string of length l with fixed ends are parallel aside through a distance d on opposite sides of a position of equilibrium, and the string is released from rest. Obtain an expression for the displacement of the string at any subsequent time and show that the midpoint of the string always remains at rest.

Solution:



To find the equation of the string into initial position OABC

The equation of the string OA is

$$\frac{x-0}{\frac{l}{3}}=\frac{y-0}{d}$$

$$\frac{3x}{l}=\frac{y}{d}$$

$$y=\frac{3xd}{l}, 0<x<\frac{l}{3}$$

The equation of the string AB is

$$\frac{x-\frac{l}{3}}{\frac{l}{3}}=\frac{y-d}{-2d}$$

$$\frac{3x-l}{l}=\frac{y-d}{-2d}$$

$$y-d=\frac{-2d}{l}(3x-l)=\frac{2d}{l}(l-3x)$$

$$y=\frac{2d}{l}(l-3x)+d=\frac{3d}{l}(l-2x), \frac{l}{3}<x<\frac{2l}{3}$$

The equation of the string BC is

$$\frac{x-\frac{2l}{3}}{l-\frac{2l}{3}}=\frac{y+d}{d}$$

$$\frac{3x-2l}{l}=\frac{y+d}{d}$$

$$y+d=\frac{d}{l}(3x-2l)$$

$$y=\frac{3xd-2dl+dl}{l}=\frac{3xd-dl}{l}=\frac{3d}{l}(x-l), \frac{2l}{3}<x<l$$

The wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions.

$$(i)\, y(0,t) = 0 \,\forall t > 0$$

$$(ii)\, y(l,t) = 0 \,\forall t > 0$$

$$(iii)\, \frac{\partial y(x,0)}{\partial t} = 0$$

$$(iv)\, y(x,0) = \begin{cases} y = \dfrac{3xd}{l}, 0 < x < \dfrac{l}{3} \\ y = \dfrac{3d}{l}(l - 2x), \dfrac{l}{3} < x < \dfrac{2l}{3} \\ y = \dfrac{3d}{l}(x - l), \dfrac{2l}{3} < x < l \end{cases}$$

The correct solution which satisfies our boundary conditions is given by

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \to (1)$$

Applying (i) in (1) we get

$$y(x,t) = c_1(c_3 \cos pat + c_4 \sin pat) = 0$$

$$(i.e)\, c_1 = 0 \;\&\; (c_3 \cos pat + c_4 \sin pat) \neq 0$$

putting $c_1 = 0$ in (1) we get

$$y(x,t) = c_2 \sin px(c_3 \cos pat + c_4 \sin pat) \to (2)$$

Applying (ii) in (2) we get

$$y(l,t) = c_2 \sin pl(c_3 \cos pat + c_4 \sin pat)$$

$$c_3 \cos pat + c_4 \sin pat \neq 0,$$

either $c_2 = 0$ or $\sin pl = 0$

Suppose if we take $c_2 = 0$ and already $c_1 = 0$ then we get a trivial solution.

$$\therefore \sin pl = 0$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Now substituting $p = \dfrac{n\pi}{l} \to (2)$ we get

$$y(x,t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l} \right) \to (3)$$

Before applying condition (iii) differentiate (3) partially w.r.to t

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left(-c_3 \sin\frac{n\pi at}{l}\cdot\frac{n\pi a}{l} + c_4 \cos\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right)$$

Applying (iii) we get

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left(c_4\frac{n\pi a}{l}\right) = 0$$

$c_2 \neq 0$ (already explained)

$$\sin\frac{n\pi x}{l} \neq 0$$

$\frac{n\pi a}{l} \neq 0$ (all are constants)

$$\therefore c_4 = 0$$

$$(3) \Rightarrow y(x,t) = c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$y(x,t) = c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (4), where c_n = c_2 c_3$$

The general solution of (4) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (5)$$

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = f(x) \rightarrow (6)$$

$$Where f(x) = \begin{cases} y = \frac{3xd}{l}, 0 < x < \frac{l}{3} \\ y = \frac{3d}{l}(l-2x), \frac{l}{3} < x < \frac{2l}{3} \\ y = \frac{3d}{l}(x-l), \frac{2l}{3} < x < l \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \rightarrow (7)$$

$$Where b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

From (6) and (7) we get $b_n = c_n$

$$c_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$c_n = \frac{6d}{l^2}\left\{\int_0^{\frac{l}{3}} x\sin\frac{n\pi x}{l}dx + \int_{\frac{l}{3}}^{\frac{2l}{3}}(l-2x)\sin\frac{n\pi x}{l}dx + \int_{\frac{2l}{3}}^{l}(x-l)\sin\frac{n\pi x}{l}dx\right\}$$

$$c_n = \frac{6d}{l^2}\left\{\left(-x\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}+\sin\frac{n\pi x}{l}\cdot\frac{l^2}{n^2\pi^2}\right)_0^{\frac{l}{3}} + \left(-(l-2x)\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}-2\sin\frac{n\pi x}{l}\cdot\frac{l^2}{n^2\pi^2}\right)_{\frac{l}{3}}^{\frac{2l}{3}} +\right.$$

$$\left.\left(-(x-l)\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}+\sin\frac{n\pi x}{l}\cdot\frac{l^2}{n^2\pi^2}\right)_{\frac{2l}{3}}^{l}\right\}$$

$$= \frac{6d}{l^2}\left\{\frac{-l^2}{3n\pi}\cos\frac{n\pi}{3}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}+\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}+\frac{l^2}{3n\pi}\cos\frac{n\pi}{3}+\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3}-\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}-\right.$$

$$\left.\frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}-\frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right\}$$

$$= \frac{6d}{l^2}\left\{\frac{3l^2}{n^2\pi^2}\sin\frac{n\pi}{3}-\frac{3l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right\}$$

$$= \frac{18d}{n^2\pi^2}\left\{\sin\frac{n\pi}{3}-\sin\frac{2n\pi}{3}\right\} = \frac{18d}{n^2\pi^2}\left\{\sin\frac{n\pi}{3}-\sin(n\pi-\frac{n\pi}{3})\right\}$$

$$= \frac{18d}{n^2\pi^2}\left(\sin\frac{n\pi}{3}-\left\{\sin n\pi\cos\frac{n\pi}{3}-\cos n\pi\sin\frac{n\pi}{3}\right\}\right) = \frac{18d}{n^2\pi^2}\sin\frac{n\pi}{3}\left[(-1)^n+1\right]$$

$$c_n = \begin{cases} \dfrac{36d}{n^2\pi^2} \quad \sin\dfrac{n\pi}{3}, niseven \\ 0, nisodd \end{cases}$$

$$y(x,t) = \frac{36d}{\pi^2} \sum_{n=2,4,6,....}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{3} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \rightarrow (8)$$

The displacement at the midpoint is got by putting $x = \frac{l}{2}$ in (8)

(i.e) $y\left(\frac{l}{2}, t\right) = 0$

There is no displacement at x=l/2

The midpoint of the string is at rest.

Problems with Zero boundary values-Temperatue or Temperature gradients

Problem

A homogeneous rod of conducting materials of length l has its ends kept at zero temperatue.The temperature at the centre is T and falls uniformly to zero at the ends. Find u(x,t).

Solution

The temperature function u(x,t) satisfies the one dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

From the given problem we get the following boundary and initial conditions

(i) $u(0,t) = 0, \forall t \geq 0$

(ii) $u(l,t) = 0, \forall t \geq 0$

Since the temperature at the centre is T and falls uniformly zero at the two ends, its distribution at t=0 is as given in the figure.

The equation of OB is

$$\frac{x - x_1}{x_2 - x_1} = \frac{x - u_1}{u_2 - u_1}$$

$$\frac{x - 0}{\frac{l}{2} - 0} = \frac{u - 0}{T - 0}$$

$$\frac{2x}{l} = \frac{u}{T}$$

$$u = \frac{2Tx}{l}, 0 \le x \le \frac{l}{2}$$

The equation BA is

$$\frac{x - x_1}{x_2 - x_1} = \frac{u - u_1}{u_2 - u_1}$$

$$\frac{x - \frac{l}{2}}{l - \frac{l}{2}} = \frac{u - T}{0 - T}$$

$$\frac{x - \frac{l}{2}}{\frac{l}{2}} = \frac{u - T}{-T}$$

$$\frac{x}{\frac{l}{2}} - 1 = -\frac{u}{T} + 1$$

$$\frac{u}{T} = 1 - \frac{x}{\left(\frac{l}{2}\right)} + 1$$

$$= 2 - \frac{2x}{l} = 2\frac{(l - x)}{l}$$

$$u = \frac{2T(l - x)}{l}, \frac{l}{2} \le x \le l$$

Hence (iii) $u(x,0) = f(x) = \begin{cases} \frac{2Tx}{l} if\, 0 \le x \le \frac{l}{2} \\ \frac{2T(l-x)}{l} if\, \frac{l}{2} \le x \le l \end{cases}$

Now the suitable solution which satisfies the boundary conditions is give by

$$u(x,t) = \left(A\cos px + B\sin px)e^{-\alpha^2 p^2 t}.\right) ....(1)$$

Applying condition (i) in equation (1) we get

$$u(0,t) = (A + 0)e^{-\alpha^2 p^2 t}. = 0$$

$Ae^{-\alpha^2 p^2 t} = 0.$

Here $e^{-\alpha^2 p^2 t}. \neq 0$ since it is defined for all t

Therefore A=0

Substitute, A=0 in equation (1) we get

$u(x,t) = (B\sin px)e^{-\alpha^2 p^2 t}$........(2)

Now applying condition (ii) in equation (2) we get

$u(l,t) = (B\sin plx)e^{-\alpha^2 p^2 t} = 0$

Here $e^{-\alpha^2 p^2 t} \neq 0$, it is defined for all t

$B \neq 0$ If B=0 already A=0 then we get a trivial solution

$\sin pl = 0$
$l = n\pi$ since $\sin n\pi = 0$

$$p = \frac{n\pi}{l}$$

Substitute, $p = \frac{n\pi}{l}$ in equation (2) we get

$$u(x,t) = B\sin\frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$u(x,t) = B\sin\frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \text{............(3)}$$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \text{......(4)}$$

Applying condition (iii) in equation (4) we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = f(x) \text{.....(5)}$$

To find B_n expand f(x) in half range Fourier sine series in the interval [0,l]

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{....(6)}$$

Where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}$

From equation (5) and (6) we get $B_n = b_n$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$B_n = \frac{2}{l}\left[\int_0^{l/2} \frac{2Tx}{l}\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} \frac{2T(l-x)}{l}\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{4T}{l^2}\left[\int_0^{l/2} x\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} (l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{4T}{l^2} x\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right] - (1)\left[\left[\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]\right]_0^{l/2}$$

$$+ \frac{4T}{l^2}[(l-x)]\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right] - (-1)\left[\left[\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]\right]_{l/2}^{l}$$

$$= \frac{4T}{l^2}\left[-x\left(\frac{l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) + \left(\frac{l}{n\pi}\right)\sin\left(\frac{n\pi x}{l}\right)\right]_0^{l/2}$$

$$+ \frac{4T}{l^2}\left[-(l-x)\left(\frac{l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) - \left(\frac{l}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{l}\right)\right]_0^{l/2}$$

$$= \frac{4T}{l^2}\left[-\left(\frac{l^2}{2n\pi}\right)\cos\left(\frac{n\pi}{2}\right)+\left(\frac{l^2}{2n\pi}\right)\cos\left(\frac{n\pi}{2}\right)+\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]_0^{l/2}$$

$$= \frac{4T}{n^2 l^2}\sin\left(\frac{n\pi}{2}\right)$$

Substitute B_n values in equation (4) we get

$$u(x,t) = \sum_{n=1}^{\infty}\frac{8T}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{l}\right)e^{\frac{-\alpha^2 n^2\pi^2 t}{l^2}}$$

$$= \frac{8T}{n^2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{l}\right)e^{\frac{-\alpha^2 n^2\pi^2 t}{l^2}}$$

1. A rod 30 cm long its ends A and B kept at 20^o and 80^o respectively until steady state conditions prevail. The temperature at each and is then suddenly reduced to $0^o C$ and kept so. Find the resulting temperature function $u(x,t)=0$ taking x=0 at A.

Solution

The temperature function $u(x,t)$ is the solution of the one dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$.......$(A)$ when the steady state condition prevails $\frac{\partial u}{\partial t}=0$ and hence we get

$\frac{\partial^2 u}{\partial x^2}=0$. Therefore (A) reduces to $\frac{d^2 u}{dx^2}=0$, on integration

$u(x) = ax+b$..........(B)

When x=0, we get $u(0)=b=20$

When x=30, we get $u(30)=30a+b$

$80=30a+20, a=2$

Thus $u(x,0)=f(x)=2x+20$ by (B)

Hence Boundary and initial condition are

(i) $u(0,t)=0, \forall t\geq 0$

(ii) $u(30,t) = 0, \forall t \geq 0$

(iii) $u(x,0) = f(x) = 2x + 20$

Now the suitable solution which statisfies our boundary conditions is given by

$$u(x,t) = \left(A\cos px + B\sin px)e^{-\alpha^2 p^2 t}.\right) \text{........(1)}$$

Applying condition (i) in equation (1) we get

$$u(0,t) = (A+0)e^{-\alpha^2 p^2 t}. = 0$$

$$Ae^{-\alpha^2 p^2 t} = 0.$$

Here $e^{-\alpha^2 p^2 t}. \neq 0$ since it is defined for all t

Therefore A=0

Substitute, A=0 in equation (1) we get

$$u(x,t) = (B\sin px)e^{-\alpha^2 p^2 t} \text{........(2)}$$

Now applying condition (ii) in equation (2) we get

$$u(30,t) = (B\sin 30p)e^{-\alpha^2 p^2 t} = 0$$

Here $e^{-\alpha^2 p^2 t} \neq 0$, it is defined for all t

$B \neq 0$ If B=0 already A=0 then we get a trivial solution

$$\begin{aligned} \sin 30p &= 0 \\ \sin 30p &= \sin n\pi \end{aligned} \quad \text{since } \sin n\pi = 0$$

$$30p = n\pi$$

$$p = \frac{n\pi}{30}$$

Substitute, $p = \dfrac{n\pi}{30}$ in equation (2) we get

$$u(x,t) = B\sin\frac{n\pi x}{30} e^{\frac{-\alpha^2 n^2 \pi^2 t}{900}} \text{..........(3)}$$

The most general solution is

$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30} e^{\frac{-\alpha^2 n^2 \pi^2 t}{900}} ......(4)$$

Applying condition (iii) in equation (4) we get

$$u(x,0)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30} = 2x+20.....(5)$$

To find B_n expand $2x+20$ in half range Fourier sine series in the interval [0,30]

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{30}....(6)$$

Where $b_n = \frac{2}{30}\int_0^l f(x)\sin\frac{n\pi x}{30}dx$

From equation (5) and (6) we get $B_n = b_n$

$$B_n = \frac{2}{30}\int_0^{30}(2x+20)\sin\frac{n\pi x}{30}dx$$

$$B_n = \frac{1}{15}\left[(2x+20)\left(\frac{-\cos\frac{n\pi x}{30}}{\left(\frac{n\pi}{30}\right)}\right)-(2)\left[\frac{\sin\frac{n\pi x}{30}}{\frac{2}{\left(\frac{n\pi}{30}\right)}}\right]\right]_0^{30}$$

$$B_n = \frac{1}{15}\left[-(2x+20)\left(\frac{30}{n\pi}\right)\cos\frac{n\pi x}{30}+(2)\left(\frac{30}{n\pi}\right)^2 \sin\frac{nx\pi}{30}\right]_0^{30}$$

$$B_n = \frac{1}{15}\left[-\left(\frac{2400}{n\pi}\cos n\pi+0\right)-\left(\frac{-600}{n\pi}\right)-0\right]$$

$$=\frac{1}{15}\left[-\frac{2400}{n\pi}\cos n\pi+\left(\frac{600}{n\pi}\right)\right]$$

$$=\frac{600}{15n\pi}[-4\cos n\pi+1]$$

$$=\frac{40}{n\pi}[1-4\cos n\pi]$$

$$= \frac{40}{n\pi}[1-4(-1)^n]$$

Substitute the value of B_n in equation (4) we get

$$u(x,t) = \frac{40}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-4(-1)^n}{n}\right]\sin\frac{n\pi x}{30}e^{\frac{-\alpha^2 n^2\pi^2 t}{900}}.$$

4.3 Steady state solution of two dimensional heat equation :

The two dimensional unsteady state heat flow equation is given by

$$\frac{\partial u}{\partial t} = \alpha^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right].$$

Type-1:

Finite plate with value given in x-direction:

Problem-:

A square plate is bounded by the lines x=0, y=0,x=10,y=10.It's faces are insulated. The temperature along the upper horizontal edge is given by u(x,0)=x(10-x) while the other three edges are krpt at

0^0C.Find the steady state temperature in the plate.

Solution:

The equation to be solved is $\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$=0.

The boundary conditions are

i)u(0,y)=0, $0 \le y \le 10$

ii) u(10,y)=0, $0 \le y \le 10$

iii) u(x,0)=0, $0 \le y \le 10$

iv)u(x,10)=x(10-x), $0 \le x \le 10$.

The suitable solution is given by

u(x,y)=(A cospx+ B sinpx) (C e^{py}+De^{-py})……………..(1)

By using the first boundary condition in (1) we get,

$u(0,y)=A(Ce^{py}+De^{-py})=0.$

Since $(Ce^{py}+De^{-py})\neq 0,$

$A=0.$

Put A=0 in (1) we get,

$u(x,y)=B\sin px\,(Ce^{py}+De^{-py})$……………….(2)

By using the second boundary condition in (2) we get,

$U(10,y)= B\sin 10p\,(Ce^{py}+De^{-py})$

Since $(Ce^{py}+De^{-py})\neq 0$, and $B\neq 0$.

$\text{Sin}10p=0$

Since $\sin n\pi=0,$

$\text{Sin}10p= \sin n\pi$

$10p= n\pi$

$P=\frac{n\pi}{10}.$

Put $p=\frac{n\pi}{10}$ in (2),

$u(x,y)=B\sin\frac{n\pi}{10}x(Ce^{\frac{n\pi}{10}y}+De^{\frac{-n\pi}{10}y})$………………(3)

By using the third boundary condition in (3) we get,

$U(x,0)=B\sin\frac{n\pi}{10}x(C+D)=0$

Since $\sin\frac{n\pi}{10}x\neq 0, B\neq 0,$

$C+D=0$

$D= -C.$

Substitute D= -C in (3),

$u(x,y)= B\sin\frac{n\pi}{10}x(Ce^{\frac{n\pi}{10}y}-Ce^{\frac{-n\pi}{10}y})$

$=BC \sin \frac{n\pi}{10} x (e^{\frac{n\pi}{10}y} - e^{\frac{-n\pi}{10}y})$

u(x,y)= $2BC \sin \frac{n\pi}{10} x \sinh \frac{n\pi}{10} y$. [since $\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$]…………(4)

The most general solution is given by

u(x,y)= $\sum_{n=0}^{\infty} A_n \sin \frac{n\pi}{10} x \sinh \frac{n\pi}{10} y$………………(5)

Apply condition (iv) in (5),

u(x,10)= $\sum_{n=0}^{\infty} A_n \sin \frac{n\pi}{10} x \sinh n\pi$ =x(10-x)

Put $B_n = A_n \sinh n\pi$, $A_n = \frac{B_n}{\sinh n\pi}$.

u(x,10)= $\sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{10} x$ =x(10-x)………………(6)

To find B_n, expand f(x) in a Fourier half range sine series.

Let $B_n = b_n$

x(10-x)= $\sum_{n=0}^{\infty} b_n \sin \frac{n\pi}{10} x$ ……………(7)

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx .$$

$$Bn = \frac{2}{10} \int_0^{10} f(x) \sin \frac{n\pi x}{10} dx$$

$$= \frac{1}{5} \int_0^{10} (10x - x^2) \sin \frac{n\pi x}{10} dx$$

$$= \frac{1}{5} \left[(10x - x^2) \left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} \right) - (10 - 2x) \left(\frac{-\sin \frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2} \right) - 2 \left(\frac{\cos \frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^3} \right) \right]_0^{10}$$

$$=\frac{1}{5}\left[-2\left(\frac{10}{n\pi}\right)^3\cos n\pi+2\left(\frac{10}{n\pi}\right)^3\right]$$

cos$n\pi$ =$(-1)^n$

$$=\frac{1}{5}\left[-2\left(\frac{10}{n\pi}\right)^3(-1)^n+2\left(\frac{10}{n\pi}\right)^3\right]$$

$$=\frac{1}{5}2\left[\frac{10}{n\pi}\right]^3\left[1-(-1)^n\right]$$

$$=\frac{2000}{5n^3\pi^3}\left[1-(-1)^n\right]$$

$$=\frac{400}{n^3\pi^3}\left[1-(-1)^n\right]$$

If n is odd $B_n=\frac{800}{n^3\pi^3}$

If n is even $B_n=0$.

Substitute the value of B_n in A_n

$A_n=\frac{800}{n^3\pi^3\sinh n\pi}$.

Substitute the value of A_n in (5)

$$u(x,y)=\sum_{n=odd}^{\infty}\frac{800}{n^3\pi^3\sinh n\pi}\sin\frac{n\pi}{10}x\sinh\frac{n\pi}{10}y.$$

Type-2

Finite plate with value given in y-direction:

Problem-:

A square plate is bounded by the lines x=0,x=a,y=0,y=a ,of a square plane bounded by the lines x=0,y=0,y=a are kept at temperature 0^0C.

The side x=a is kept at temperature given by u(a,y)=100,0<y<a.

Solution:

The equation to be solved is $\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]=0.$

The boundary conditions are

i)u(0,y)=0,

ii) u(x,a)=0,

iii) u(0,y)=0,

iv)u(a,y)=100.

The suitable solution is given by

u(x,y)= (A e^{px}+Be^{-px}) (C cospy+ D sinpy) ……………….(1)

By using the first boundary condition in (1) we get,

u(x,0)= (A e^{px}+Be^{-px}) C=0

(A e^{px}+Be^{-px}) $\neq 0$

C=0.

Put C=0 in (1) we get,

u(x,y)= (A e^{px}+Be^{-px}) (D sinpy)……………(2)

By using the second boundary condition in (2) we get,

u(x,a)= (A e^{px}+Be^{-px}) (D sinpa)=0

Here (A e^{px}+Be^{-px}) $\neq 0$,D$\neq 0$,

P=$\frac{n\pi}{a}$

Put p=$\frac{n\pi}{a}$ in (2),

u(x,y)= $\left(Ae^{\frac{n\pi x}{a}} + Be^{\frac{-n\pi x}{a}} \right)$(D sin$\frac{n\pi y}{a}$)……………….(3)

By using the third boundary condition in (3) we get,

u(0,y)=(A+B)Dsin$\frac{n\pi y}{a}$ =0

Here sin$\frac{n\pi y}{a} \neq 0$, D$\neq 0$,

A+B=0, B= -A

Put B=-A in (3) we get,

u(x,y)= $\left(Ae^{\frac{n\pi x}{a}} - Ae^{\frac{-n\pi x}{a}} \right)$D $\sin\frac{n\pi y}{a}$

$$= AD\left(e^{\frac{n\pi x}{a}} - e^{\frac{-n\pi x}{a}} \right)\sin\frac{n\pi y}{a}$$

$$= 2AD\sinh\frac{n\pi x}{a}\sin\frac{n\pi y}{a}......................(4)$$

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi x}{a}\sin\frac{n\pi y}{a}..................(5)$$

Applying condition (4) in (5),

$$u(a,y) = \sum_{n=1}^{\infty} A_n \sinh n\pi \sin\frac{n\pi y}{a} = 100$$

$$u(a,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi y}{a} = 100....................(6)$$

To find B_n expand 100 in a Fourier half range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{a}..................(7) where \quad b_n = \frac{2}{l}\int_0^l 100\sin\frac{n\pi y}{a}dy$$

From (6) and (7) B_n=b_n.

$$B_n = \frac{2}{a}\int_0^a 100\sin\frac{n\pi y}{a}dy$$

$$= \frac{200}{a}\left[\frac{-\cos\frac{n\pi y}{a}}{\frac{n\pi}{a}}\right]_0^a$$

$$= \frac{-200}{n\pi}\left[(-1)^n - 1\right]$$

Bn=0 if n is even.

$$B_n = \frac{400}{n\pi} ifnisodd$$

An=0if n is even.

$$A_n = \frac{400}{n\pi \sinh \pi} ifnisodd$$

Substitute the value of An in (5)

$$u(x,y) = \sum_{n=odd}^{\infty} \frac{400}{n\pi \sinh n\pi} \sinh \frac{n\pi x}{a} \sin \frac{n\pi y}{a}.$$

Type-3:

Infinite Plates:

1.A rectangular Plate with insulated surface is 10 cm wide ,so long compared to its with that it may be considered infinite in length. If the temperature at the short edge y = 0 is given by u = x for $0 \le x \le 5$ and $10 - x$ for $5 \le x \le 10$ and the two long edges x = 0 , x= 10 as well as the other short edge are kept at 0^0c.Find the temperature function u(x,y) at any point of the plate.

solution:

The equation to be solved is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

From the given problem we get the following boundary conditions.

1.u(0,y) = 0, for all y

2.u(10,y) = 0, for all y

3.u(x, ∞) = 0,

4.u(x,0) = $\begin{cases} x, 0 \le x \le 5 \\ 10 - x, 5 \le x \le 10 \end{cases}$

now the suitable solution which satisfies our boundary conditions is given by

$u(x,y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py})$……….(1)

Applying condition (1) in (1),we get $u(0,y) = 0 \Rightarrow A(Ce^{py} + De^{-py}) = 0$

here $Ce^{py} + De^{-py} \neq 0$

$\therefore A = 0$

substitute $A = 0$ in (1),we get $u(x,y) = B \sin px(Ce^{py} + De^{-py})$………(2)

Applying condition (2) in (2),we get $u(10,y) = 0 \Rightarrow u(10,y) = B \sin 10p(Ce^{py} + De^{-py}) = 0$

here $Ce^{py}+De^{-py}\neq 0$

$B\neq 0$

$\therefore \sin 10p=0$

$10p=n\pi$

$p=\dfrac{n\pi}{10}$

substitute $p=\dfrac{n\pi}{10}$ in(2),we get $u(x,y)=B\sin\dfrac{n\pi x}{10}\left(Ce^{\frac{n\pi y}{10}}+De^{\frac{-n\pi y}{10}}\right)$……….(3)

Applying condition (3) in (3),we get $u(x,\infty)=0\Rightarrow B\sin\dfrac{n\pi x}{10}\left(Ce^{\infty}+De^{-\infty}\right)=0$

$B\sin\dfrac{n\pi x}{10}Ce^{\infty}=0$, $\qquad \because e^{-\infty}=0$

here $B\neq 0$

$\sin\dfrac{n\pi x}{10}\neq 0$

$e^{\infty}\neq 0$

$\therefore C=0$

substitute $C=0$ in(3),we get $u(x,y)=B\sin\dfrac{n\pi x}{10}De^{\frac{-n\pi y}{10}}$

$u(x,y)=BD\sin\dfrac{n\pi x}{10}e^{\frac{-n\pi y}{10}}$ ………..(4)

The most general solution is $u(x,y)=\sum_{1}^{\infty}B_n\sin\dfrac{n\pi x}{10}e^{\frac{-n\pi y}{10}}$ ………..(5)

Now applying condition (4) in (5),we get $u(x,0)=\sum_{1}^{\infty}B_n\sin\dfrac{n\pi x}{10}=\begin{cases}x, 0\leq x\leq 5\\ 10-x, 5\leq x\leq 10\end{cases}$ ……….(6)

To find B_n then we expand f(x) as a Fourier half range sine series in (0,10)

$f(x)=\sum_{1}^{\infty}B_n\sin\dfrac{n\pi x}{10}$ ………..(7) when $b_n=\dfrac{2}{10}\int_0^{10}f(x)\sin\dfrac{n\pi x}{10}dx$

From (6) & (7),we get $B_n = b_n$

$$B_n = \frac{2}{10}\left[\int_0^5 x\sin\frac{n\pi x}{10}dx + \int_5^{10}(10-x)\sin\frac{n\pi x}{10}dx\right]$$

$$=\frac{1}{5}\left[\left[x\left(\frac{-\cos\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{10}}{(\frac{n\pi}{10})^2}\right)\right]_0^5+\left[10-x\left(\frac{-\cos\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)}\right)-(-1)\left(\frac{-\sin\frac{n\pi x}{10}}{(\frac{n\pi}{10})^2}\right)\right]_5^{10}\right]$$

$$=\frac{1}{5}\left[-x\left(\frac{10}{n\pi}\right)\cos\frac{n\pi x}{10}+\left(\frac{10}{n\pi}\right)^2\sin\frac{n\pi x}{10}\right]_0^5+\frac{1}{5}\left[-(10-x)\left(\frac{10}{n\pi}\right)\cos\frac{n\pi x}{10}-\left(\frac{10}{n\pi}\right)^2\sin\frac{n\pi x}{10}\right]_5^{10}$$

$$=\frac{1}{5}\left[-\left(\frac{50}{n\pi}\right)\cos\frac{n\pi}{2}+\left(\frac{10}{n\pi}\right)^2\sin\frac{n\pi}{2}-(0+0)\right]+\frac{1}{5}\left[(-0-0)-\left(\left(\frac{-50}{n\pi}\right)\cos\frac{n\pi}{2}-\left(\frac{10}{n\pi}\right)^2\sin\frac{n\pi}{2}\right)\right]$$

$$=\frac{1}{5}\left[-\frac{50}{n\pi}\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}+\left(\frac{50}{n\pi}\right)\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{1}{5}\left[\frac{200}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{40}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$B_n = \frac{40}{n^2\pi^2}\sin\frac{n\pi}{2}$ =0 if n is even

substitute the value of B_n in equation (5) we get

$$u(x,y) = \sum_{n=0dd}^{\infty}\frac{40}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}e^{\frac{-n\pi y}{10}}$$

$$\therefore u(x,y) = \frac{40}{\pi^2}\sum_{n=0dd}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}e^{\frac{-n\pi y}{10}}.$$

UNIT-5

Z-Transforms and Difference Equations

5.1-Z-Transform,elementary properties of Z-transform.

Z-Transform (Unilateral):

Let {x(n)} be a sequence defined for n=0,1,2,….. and x(n)=0 for n<0, then its Z-Transform is defined to be Z{x(n)}=X(z)= $\sum_{n=0}^{\infty} x(n)z^{-n}$, where Z is an arbitrary complex number.

Z-Transform (Bilateral):

Let {x(n)} be a sequence defined for all integers then its Z-Transform is defined to be Z{x(n)}=X(z)= $\sum_{n=-\infty}^{\infty} x(n)z^{-n}$, where Z is an arbitrary complex number.

Problems based on Z-transform of some basic functions:

Problem: Prove that z[1] = $\frac{z}{z-1}, |z| > 1$.

Solution:

$$Z\{x(n)\}=\sum_{n=0}^{\infty} x(n)z^{-n}$$

Since x(n)=1,

$$z[1]= \sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$=1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\ldots\ldots$$

$$=\left[1-\frac{1}{z}\right]^{-1} \quad [\text{Since } (1-x)^{-1}=1+x+x^2+.....]$$

$$=\left[\frac{z-1}{z}\right]^{-1}$$

$$=\left[\frac{z}{z-1}\right]$$

Problem: Prove that $z[a^n] = \frac{z}{z-a}$ if $|z| > |a|$

Solution:

We know that $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$

Since $x(n) = a^n$,

$$z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n \frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty} a^n \left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \ldots\ldots$$

$$= \left[1 - \frac{a}{z}\right]^{-1}$$

$$= \left[\frac{z-a}{z}\right]^{-1}$$

$$= \left[\frac{z}{z-a}\right]$$

Problem: Prove that $z[n] = \frac{z}{(z-1)^2}$, $|z| > 1$.

Solution:

We know that $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$

Since $x(n) = n$,

$$z[n] = \sum_{n=0}^{\infty} n z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{n}{z^n}$$

$$= \sum_{n=0}^{\infty} n \left(\frac{1}{z}\right)^n$$

$$= \frac{1}{z} + 2\left(\frac{1}{z}\right)^2 + \ldots\ldots$$

$$\frac{1}{z}$$

$$=\frac{1}{z}\left[1+2\left(\frac{1}{z}\right)+3\left(\frac{1}{z}\right)^{2}+......\right]$$

$$=\frac{1}{z}\left[\left(1-\frac{1}{z}\right)^{-2}\right].$$ [since $(1-x)^{-2} = 1+2x+3x^2+......$]

$$=\frac{1}{z}\left[\left(\frac{z-1}{z}\right)^{-2}\right]$$

$$=\frac{1}{z}\left[\left(\frac{z}{z-1}\right)^{2}\right]$$

$$=\frac{z}{(z-1)^{2}}.$$

Problem: Prove that $z\left(\frac{1}{n}\right)=\log\left(\frac{z}{z-1}\right)$ if $|z|>1, n>0$.

Solution:

We know that $Z\{x(n)\}=\sum_{n=0}^{\infty} x(n)z^{-n}$

Since $x(n)=\frac{1}{n}$,

$$z\left(\frac{1}{n}\right)=\sum_{n=0}^{\infty}\left(\frac{1}{n}\right)z^{-n}$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{nz^{n}}\right)$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{n}\right)\left(\frac{1}{z}\right)^{n}$$

$$=\frac{1}{z}+\frac{1}{2}\left(\frac{1}{z}\right)^{2}+......$$

$$= -\log\left[\left(1-\frac{1}{z}\right)\right]$$ [since $-\log(1-x)= x+\frac{1}{2}x^{2}+....$

$$= -\log\left[\left(\frac{z-1}{z}\right)\right]$$ [Since $a \log b= \log b^{a}$]

$$= \log\left[\left(\frac{z-1}{z}\right)\right]^{-1}$$

$$= \log\left[\left(\frac{z}{z-1}\right)\right]$$

Problem: Prove that z $\left[\frac{1}{n!}\right] = e^{\frac{1}{z}}$.

Solution:

We know that z{x(n)}= $\sum_{n=0}^{\infty} x(n) z^{-n}$

We have x(n)= $\left[\frac{1}{n!}\right]$

$$z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^{n}$$

$$= \mathbf{1} + \left(\frac{1}{1!}\right)\left(\frac{1}{z}\right) + \left(\frac{1}{2!}\right)\left(\frac{1}{z}\right)^{2} + \ldots\ldots$$

$$= \mathbf{1} + \frac{\left(\frac{1}{z}\right)}{(1!)} + \frac{\left(\frac{1}{z}\right)^{2}}{(2!)} + \ldots\ldots$$

$= e^{\frac{1}{z}}$. [since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\ldots$].

Problem: Find z $\left[\frac{1}{(n+1)!}\right]$.

Solution:

We know that z{x(n)}= $\sum_{n=0}^{\infty} x(n) z^{-n}$

We have x(n)= $\left[\frac{1}{(n+1)!}\right]$.

$$z\left[\frac{1}{(n+1)!}\right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\frac{1}{z}\right)^{n}$$

$$= 1 + \frac{\left(\frac{1}{z}\right)^{2}}{(2!)} + \frac{\left(\frac{1}{z}\right)^{3}}{(3!)} + \ldots\ldots$$

$$= z\left[1 + \frac{\left(\frac{1}{z}\right)}{(1!)} + \frac{\left(\frac{1}{z}\right)^{2}}{(2!)} + \ldots\ldots\right]$$

$$=z[e^{\frac{1}{z}}-1]. \text{ [since } e^x - 1 = \frac{x}{1!}+\frac{x^2}{2!}+........\text{]}.$$

Problem: Find the z-transform of (n+2).
Solution:

$$Z(n+2)=Z(n)+z(2)$$
$$=z(n)+2z(1)$$
$$=\frac{z}{(z-1)^2}+\frac{2z}{(z-1)} \text{ [since } z(n)=\frac{z}{(z-1)^2}, z(1)=\frac{z}{(z-1)}\text{]}.$$

Problems based on Bilateral Z-transform

Problem: Find z $\{a^{|n|}\}$.
Solution:

$$z\{x(n)\}=\sum_{n=-\infty}^{\infty} x(n)z^{-n}$$
$$x(n)=\{a^{|n|}\}.$$
$$z\{a^{|n|}\}=\sum_{n=-\infty}^{\infty} a^{|n|}z^{-n}.$$
$$=\sum_{n=-\infty}^{-1} a^{-n}z^{-n}+\sum_{n=0}^{\infty} a^{n}z^{-n}.$$
$$=[.....+a^3z^3+a^2z^2+az]+z[a^n]$$
$$=\frac{az}{1-az}+\frac{z}{z-a}$$
$$=\frac{z-a^2z}{(1-az)(z-a)}.$$

Problems based on Z(1) and Z(a^n)

Problem: Find z[e^{-an}]
Solution:

$$Z(a^n)=\frac{z}{z-a}$$
$$z[e^{-an}]=z[(e^{-a})^n]$$
$$=\frac{z}{z-e^{-a}}.$$

Problem: Find z[cos nθ] and z[sin nθ].
Solution:

Let $a = e^{i\theta}$
$$A^n=(e^{i\theta})^n$$
$$=e^{in\theta}$$

=cos nθ+i sin nθ

We know that Z(a^n)= $\frac{z}{z-a}$.

Z[$\left(e^{i\theta}\right)^n$]= $\frac{z}{z-e^{i\theta}}$.

Z[$e^{in\theta}$]= $\frac{z}{z-(\cos\theta+i\sin\theta)}$.

$$=\frac{z}{z-(\cos\theta+i\sin\theta)}$$

Z[cosnθ]+iz[sinnθ]= $\left[\frac{z}{(z-\cos\theta)-i\sin\theta}\right]\left[\frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}\right]$

$$=\frac{z(z-\cos\theta)+iz\sin\theta}{(z-\cos\theta)^2+\sin^2\theta}$$

$$=\frac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$=\frac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+1}$$

$$=\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}+\frac{iz\sin\theta}{z^2-2z\cos\theta+1}$$

Equating the real and imaginary parts,

Z{cosnθ}= $\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

Z{sinnθ}= $\frac{z\sin\theta}{z^2-2z\cos\theta+1}$.

Problem: Find z[cos an]

Solution:

Put θ=a in problem 10,

We get

Z{cosnθ}= $\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

Problem: Find z[$(-1)^n$]

Solution:

Z[a^n**]=** $\frac{z}{z-a}$

z[$(-1)^n$]= $\frac{z}{z-(-1)}$

$$=\frac{z}{z+1}.$$

Problem: Find $z\left[\cos\frac{n\pi}{2}\right]$.

Solution:

$$z\left[\cos\frac{n\pi}{2}\right]=\frac{z\left[z-\cos\frac{\pi}{2}\right]}{z^2-2z\cos\frac{\pi}{2}+1}$$

$$=\frac{z[z-0]}{z^2-2z(0)+1}$$

$$=\frac{z^2}{z^2+1}.$$

Problem: Find z $\left[\frac{1}{n(n+1)}\right]$.

Solution:

$$\left[\frac{1}{n(n+1)}\right]=\frac{A}{n}+\frac{B}{n+1}$$

1=A(n+1)+B(n)

Put n=0 ,A=1.

Put n= -1, B= -1.

$$\left[\frac{1}{n(n+1)}\right]=\frac{1}{n}-\frac{1}{n+1}.$$

Take z on both sides,

$$z\left[\frac{1}{n(n+1)}\right]=z\left[\frac{1}{n}\right]-z\left[\frac{1}{n+1}\right]$$

We know that $z\left[\frac{1}{n}\right]=\log\frac{z}{z-1}$,$z\left[\frac{1}{n+1}\right]=z\log\frac{z}{z-1}$.

$$z\left[\frac{1}{n(n+1)}\right]=\log\frac{z}{z-1}-z\log\frac{z}{z-1}.$$

$$=(1-z)\log\frac{z}{z-1}.$$

Problem: Find z $\left[\frac{1}{(n+1)(n+2)}\right]$.

Solution:

$$\left[\frac{1}{(n+1)(n+2)}\right]=\frac{A}{n+1}+\frac{B}{n+2}$$

Put n=-1,A=1.

Put n= -2,B= -1.

$$\left[\frac{1}{(n+1)(n+2)}\right]=\frac{1}{n+1}-\frac{1}{n+2}.$$

$$z\left[\frac{1}{(n+1)(n+2)}\right]=z\left[\frac{1}{n+1}\right]-z\left[\frac{1}{n+2}\right]$$

$$= z \log \frac{z}{z-1} - \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}$$

$$= z \log \frac{z}{z-1} - \left[\frac{1}{2} + \frac{1}{3}\left(\frac{1}{z}\right) + \frac{1}{4}\left(\frac{1}{z}\right)^2 + ... \right]$$

$$= z \log \frac{z}{z-1} - z^2 \left[\frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + ... \right]$$

$$= z \log \frac{z}{z-1} + z^2 \log\left[\frac{z-1}{z}\right] + z$$

$$= z \log \frac{z}{z-1} - z^2 \log\left[\frac{z}{z-1}\right] + z$$

$$=(z-z^2)\log\left[\frac{z}{z-1}\right] + z.$$

Problem: Find the z-transform of $f(n)=\frac{2n+3}{(n+1)(n+2)}$.

Solution:

$$\frac{2n+3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

2n+3=A(n+2)+B(n+1).

Put n= -1 ,A=1.

Put n= -2,B=1.

$$\frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}$$

Take z on both sides,

$$z\left[\frac{2n+3}{(n+1)(n+2)}\right] = z\left[\frac{1}{n+1}\right] + z\left[\frac{1}{n+2}\right]$$

We know that,

$$z\left[\frac{1}{n+1}\right] = z \log \frac{z}{z-1} \quad \text{……….(1)}$$

$$z\left[\frac{1}{n+2}\right] = \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}.$$

$$= \frac{1}{2} + \frac{1}{3}\left(\frac{1}{z}\right) + \frac{1}{4}\left(\frac{1}{z}\right)^2 + …….$$

$$= z^2 \left[\frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + ... \right]$$

$$= z^2 \left[-\log\left(1 - \frac{1}{z}\right) - \frac{1}{z} \right]$$

$$= z^2\left[\log\left(\frac{z}{z-1}\right) - z\right] \text{..............(2)}$$

By 1 and 2 ,

$$Z[f(n)] = z\log\frac{z}{z-1} + z^2\left[\log\left(\frac{z}{z-1}\right) - z\right]$$

$$= z\log\frac{z}{z-1}[1+z] - z.$$

First shifting theorem:

i)If $z\{f(t)\} = F(z)$,then $z\{e^{-at}f(t)\} = F[ze^{aT}]$.

ii)If $z\{f(t)\} = F(z)$,then $z\{e^{at}f(t)\} = F[ze^{-aT}]$.

Damping rule:

iii) If $z\{f(t)\} = F(z)$,then $z\{a^n f(t)\} = F\left[\frac{z}{a}\right]$.

iv) If $z\{f(n)\} = F(z)$,then $z\{a^n f(n)\} = F\left[\frac{z}{a}\right]$.

Problems based on first shifting theorem:

Problem: Find $z[a^n n]$.

Solution:

We know that $z\{a^n f(n)\} = F\left[\frac{z}{a}\right]$.

$$z[a^n n] = [z(n)]_{z\to\frac{z}{a}}$$

$$= \left[\frac{z}{(z-1)^2}\right]_{z\to\frac{z}{a}} \quad \left[\text{since } z(n) = \left[\frac{z}{(z-1)^2}\right]\right.$$

$$= \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2}$$

$$= \frac{\frac{z}{a}}{\left(\frac{z-a}{a}\right)^2}$$

$$= \frac{az}{(z-a)^2}$$

Problem: Find $z\left[\frac{a^n}{n!}\right]$

Solution:

We know that $z\{a^n f(n)\} = F\left[\frac{z}{a}\right]$.

$$z\left[\frac{a^n}{n!}\right] = \left[z\left(\frac{1}{n!}\right)\right]_{z\to\frac{z}{a}}$$

$$= \left[e^{\frac{1}{z}}\right]_{z\to\frac{z}{a}}$$

$$= e^{\frac{a}{z}}.$$

Differentiation in the Z-Domain:

i) $Z[nf(t)] = -z\,\frac{d}{dz}F(z)$

ii) $Z[nf(n)] = -z\,\frac{d}{dz}F(z)$

Proof:

i) Given $F(z) = z[f(t)]$

$$F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

$$\frac{d}{dz}[F(z)] = \sum_{n=0}^{\infty} -nf(nT)z^{-n-1}$$

$$= \sum_{n=0}^{\infty} -nf(nT)\frac{z^{-n}}{z}$$

$$z\frac{dF}{dz} = \sum_{n=0}^{\infty} -nf(nT)z^{-n}$$

$$= -z[nf(t)]$$

$$Z[nf(t)] = -z\frac{d}{dz}[F(z)].$$

ii) Given $F(z) = z[f(n)]$

$$F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}$$

$$\frac{d}{dz}[F(z)] = \sum_{n=0}^{\infty} -nf(n)z^{-n-1}$$

$$= \sum_{n=0}^{\infty} -nf(n)\frac{z^{-n}}{z}$$

$$z\frac{dF}{dz} = \sum_{n=0}^{\infty} -nf(n)z^{-n}$$

$$= -z[nf(n)]$$

$$Z[nf(n)] = -z\frac{d}{dz}[F(z)].$$

Problem: Find z(n^2)

Solution:

We know that Z[nf(n)]= $-z\frac{d}{dz}$ [F(z)].

z(n^2)=z(n n)

$$= -z\frac{d}{dz}[z(n)].$$

$$=-z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$=-z\left[\frac{(z-1)^2 - z[2(z-1)]}{(z-1)^4}\right] \quad \left[\text{ since } d\left[\frac{u}{v}\right]=\frac{vu' - uv'}{v^2}\right]$$

$$=-z\left[\frac{z-1-2z}{(z-1)^3}\right]$$

$$=-z\left[\frac{-1-z}{(z-1)^3}\right]$$

$$=z\left[\frac{z+1}{(z-1)^3}\right]$$

$$=\left[\frac{z^2+z}{(z-1)^3}\right].$$

Problem: Find the z-transform of (n+1)(n+2)

Solution:

Z[(n+1)(n+2)]=z[n^2+2n+n+2]

=z[n^2+3n+2]

=z[n^2]+3z[n]+2z[1]

$$=\left[\frac{z^2+z}{(z-1)^3}\right]+3\left[\frac{z}{(z-1)^2}\right]+2\left[\frac{z}{(z-1)}\right]$$

$$=\left[\frac{z^2+z+3z(z-1)+2z(z-1)^2}{(z-1)^3}\right]$$

$$=\left[\frac{2z^3}{(z-1)^3}\right].$$

Second shifting theorem:

If z[f(t)]=F(z) then z[f(t+T]=zF(z)-zf(0).

Proof:

$$Z[f(t+T)]= \sum_{n=0}^{\infty} f(nT+T)z^{-n}$$

$$=\sum_{n=0}^{\infty} f(n+1)Tz^{-n}$$

$$=z\sum_{n=0}^{\infty} f(n+1)Tz^{-(n+1)}$$

Put n+1=m

$$=z\sum_{m=1}^{\infty} f(mT)z^{-m}$$

$$=z\ [\sum_{m=0}^{\infty} f(mT)z^{-m} - f(0)]$$

$$=zF(z)-f(0)$$

Problems based on second shifting theorem:

Problem: Find $z[e^{3(t+T)}]$.

Solution:

We know that z[f(t+T]=zF(z)-zf(0).

z[f(t+T]=z Z[f(t)]-zf(0).

We have $f(t)=e^{3t}, f(0)=e^0=1$

$$Z[f(t)]=Z[e^{3t}]=\frac{z}{z-e^{3T}}$$

$$z[e^{3(t+T)}]= zF(z)-zf(0).$$

$$=z\frac{z}{z-e^{3T}} - z$$

$$=z\left[\frac{z}{z-e^{3T}}-1\right]$$

$$=z\left[\frac{z-z+e^{3T}}{z-e^{3T}}\right]$$

$$=\left[\frac{ze^{3T}}{z-e^{3T}}\right].$$

Problem: Finf Z[sinh(t+T)]

Solution:

We know that $\sinh\theta=\frac{e^{\theta}-e^{-\theta}}{2}$.

$$\text{Sin}(t+T)= \frac{e^{(t+T)}-e^{-(t+T)}}{2}$$

$$Z[\sinh(t+T)]=\frac{1}{2} Z[e^{(t+T)}]- \frac{1}{2} Z[e^{-(t+T)}]$$

$$=\frac{1}{2} Z[e^{t}e^{T}]- \frac{1}{2} Z[e^{-t}e^{-T}]$$

$$=\frac{1}{2} e^{T}Z[e^{t}]= \frac{1}{2} e^{-T}Z[e^{-t}]$$

$$=\frac{1}{2}\text{eT}\frac{z}{z-e^{T}}-\frac{1}{2}\text{e-T}\frac{z}{z-e^{-T}}$$

$$=\frac{z}{2}\left[\frac{e^{T}}{z-e^{T}}-\frac{e^{-T}}{z-e^{-T}}\right]$$

$$=\frac{z}{2}\left[\frac{ze^{T}-1-ze^{-T}+1}{z^{2}-ze^{-T}-ze^{T}+1}\right]$$

$$=\frac{z^{2}}{2}\left[\frac{e^{T}-e^{-T}}{z^{2}-z(e^{T}+e^{-T})+1}\right]$$

$$=\frac{z^{2}}{2}\left[\frac{2\sin T}{z^{2}-2z\cos T+1}\right] \quad [\because \sin\theta=\frac{e^{\theta}-e^{-\theta}}{2} \quad \text{and} \quad \cos\theta=\frac{e^{\theta}+e^{-\theta}}{2}]$$

$$=z^{2}\left[\frac{2\sin T}{z^{2}-2z\cos T+1}\right].$$

Unit impulse sequence and unit step sequence

The unit impulse sequence δ (n) is defined as the sequence with values

$$\delta(n)=\begin{cases}1 \; for n=0\\ 0 \; for n\neq 0\end{cases}$$

Problems based on Unit impulse sequence and unit step sequence

Problem: Find $z[2^{n}\delta(n-2)]$

Solution:

$$z[2^{n}\delta(n-2)]= z[\delta(n-2)]_{z\to\frac{z}{2}}$$

$$=\left[\frac{1}{z^{2}}\right]_{z\to\frac{z}{2}}$$

$$=\left[\frac{1}{\left(\frac{z}{2}\right)^{2}}\right]$$

$$=\frac{4}{z^{2}}.$$

Problem: Find $z\left[\cos\frac{n\pi}{2}u(n)\right]$

Solution:

$$z\left[\cos\frac{n\pi}{2}u(n)\right]= z\left[\cos\frac{n\pi}{2}\right]$$

$$=\sum_{n=0}^{\infty}\cos\frac{n\pi}{2}z^{-n}$$

$$=1+\frac{1}{z}+\frac{1}{z^2}+.......$$

$$=\left[1-\frac{1}{z}\right]^{-1}$$

$$=\left[\frac{z-1}{z}\right]^{-1}$$

$$=\left[\frac{z}{z-1}\right].$$

Initial value theorem:

If z[f(t)]=F(z),then f(0)= $\lim_{z\to\infty}$ F(z).

Proof:

$$F(z)=z[f(t)]=\sum_{n=0}^{\infty} f(nT)z^{-n}$$

$$=f(0.T)+\frac{f(1.T)}{z}+\frac{f(2.T)}{z^2}+.......$$

$$=f(0)+\frac{f(T)}{z}+\frac{f(2.T)}{z^2}+.......$$

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty}\left[f(0)+\frac{f(T)}{z}+\frac{f(2.T)}{z^2}+.......\right]$$

=f(0).

Final value theorem:

If z[f(t)]=F(z),then $\lim_{t\to\infty}$ f(t)= $\lim_{z\to 1}$ (z-1)F(z).

Proof:

$$\mathbf{Z[f(t+T)-f(t)]}=\sum_{n=0}^{\infty} [f(nT+T)-f(nT)]z^{-n}$$

$$Z[f(t+T)]-Z[f(t)]=\sum_{n=0}^{\infty} [f(nT+T)-f(nT)]z^{-n}$$

$$zF(z)-zf(0)-F(z)=\sum_{n=0}^{\infty} [f(nT+T)-f(nT)]z^{-n}$$

Taking limit as z $\to 1$,

$$\lim_{z\to 1} [(z-1)F(z)-zf(0)]=\lim_{z\to 1}\left[\sum_{n=0}^{\infty} [f(nT+T)-f(nT)]z^{-n}\right]$$

$$\lim_{z\to 1} [(z-1)F(z)-f(0)]=\sum_{n=0}^{\infty} [f(nT+T)-f(nT)]$$

$$=\lim_{n\to\infty} [f(T)-f(0)-f(2T)-f(T)+\ldots\ldots+f[(n+1)T-f(nT)]$$

$$= \lim_{n \to \infty} f[(n+1)T]-f(0)$$

$$\lim_{z \to 1} (z-1)F(z)-f(0)= f(\infty)-f(0)$$

$$\lim_{z \to 1} (z-1)F(z)= f(\infty)$$

$$= \lim_{t \to \infty} f(t).$$

Problems based on Initial and Final value theorem:

Problem-25:

If $F(z)=\dfrac{z(z-\cos aT)}{z^2-2z\cos aT+1}$, find f(0) also find $\lim_{t \to \infty} f(t)$.

Solution:

By Initial value theorem $f(0)= \lim_{z \to \infty} F(z)$.

$$= \lim_{z \to \infty} \frac{z(z-\cos aT)}{z^2-2z\cos aT+1} = \frac{\infty}{\infty}$$

By L'Hospital rule differentiating w.r.to z,

$$= \lim_{z \to \infty} \frac{z+(z-\cos aT)}{2z-2\cos aT}$$

$$= \lim_{z \to \infty} \frac{(2z-\cos aT)}{2z-2\cos aT}$$

$$= \frac{\infty}{\infty}$$

By L'Hospital rule differentiating w.r.to z,

$$= \lim_{z \to \infty} \frac{2}{2}$$

=1.

By Final value theorem,

$$\lim_{t \to \infty} f(t)= \lim_{z \to 1} (z-1)F(z).$$

$$= \lim_{z \to 1} (z-1) \frac{z(z-\cos aT)}{z^2-2z\cos aT+1}$$

=0

By L'Hospital rule differentiating w.r.to z,

$$= \lim_{z \to 1} \frac{(z-1)[z+(z-\cos aT)]+z(z-\cos aT)}{2z-2\cos aT}$$

$$= \frac{(1-\cos aT)}{2-2\cos aT}$$

$$=\frac{(1-\cos aT)}{2(1-\cos aT}$$

$$=\frac{1}{2}.$$

5.2-Inverse Z-Transform:
The inverse Z^{-1}Transform of Z[x(n)]=X(z) is defined as Z^{-1}[x(z)]=[x(n)].

Expansion method:

Problem: Find $Z^{-1}\left[\frac{z}{z-a}\right]$

Solution:

Let X(z)= $\left[\frac{z}{z-a}\right]$

$$=\frac{z}{z\left(1-\frac{a}{z}\right)}$$

$$=\left(1-\frac{a}{z}\right)^{-1}$$

$$=1+\left(\frac{a}{z}\right)+\left(\frac{a}{z}\right)^{2}+.......$$

$$=\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^{n}$$

$$=\sum_{n=0}^{\infty}a^{n}z^{-n}$$

Coefficient of z^{-n} is a^n.

$Z^{-1}[x(z)]=a^n$

$Z^{-1}\left[\frac{z}{z-a}\right]=a^n$

Power series method:

Problem: Find $Z^{-1}\left[\log\left(\frac{z}{z+1}\right)\right]$

Solution:

Let F(z)= $\left[\log\left(\frac{z}{z+1}\right)\right]$

Put z =1/y

$$=\left[\log\left(\frac{\frac{1}{y}}{\frac{1}{y}+1}\right)\right]$$

$$=\left[\log\left(\frac{\frac{1}{y}}{\frac{1+y}{y}}\right)\right]$$

$$=\left[\log\left(\frac{1}{1+y}\right)\right]$$

$=\log[(1+y)]^{-1}$

$=-\log(1+y)$

$=-y+\frac{1}{2}(y^2)-\frac{1}{3}(y^3)+\ldots$

$$=-\frac{1}{z}+\frac{1}{2z^2}-\frac{1}{3z^3}+\ldots+\frac{(-1)^n}{n}z^{-n}$$

$F(n)=Z^{-1}[f(z)]$

$$=\left\{\begin{matrix} 0 \; for n = 0 \\ \frac{(-1)^n}{n} \; otherwise \end{matrix}\right\}.$$

Partial Fractions Method:

Problem: Find $Z^{-1}\left[\left(\frac{10z}{(z-1)(z-2)}\right)\right]$

Solution:

Let $X(z)=\left[\left(\frac{10z}{(z-1)(z-2)}\right)\right]$

$$\frac{X(z)}{z}=\frac{10}{(z-1)(z-2)}$$

$$\frac{X(z)}{z}=\frac{A}{z-1}+\frac{B}{z-2} \ldots\ldots\ldots\ldots(1)$$

10=A(z-2)+B(z-1)

Put z=1,A= -10

Putz=2,B= 10.

Substitute the values of A and B in (1)

$$\frac{X(z)}{z}=\frac{-10}{z-1}+\frac{10}{z-2}$$

$$X(z)=\frac{-10z}{z-1}+\frac{10z}{z-2}$$

$Z\{x(n)=10\left[\frac{z}{z-2}\right]-10\left[\frac{z}{z-1}\right]$

$X(n)=10Z^{-1}\left[\frac{z}{z-2}\right]-10Z^{-1}\left[\frac{z}{z-1}\right]$

$=10(2^n)-10$

$=10(2^n-1)$

Problem-29:

Find $Z^{-1}\left[\left(\frac{z(z^2-z+2)}{(z+1)(z-1)^2}\right)\right]$

Solution:

Let $X(Z)=\left[\left(\frac{z(z^2-z+2)}{(z+1)(z-1)^2}\right)\right]$

$\frac{X(z)}{z}=\left[\left(\frac{(z^2-z+2)}{(z+1)(z-1)^2}\right)\right]$

$\frac{X(z)}{z}=\frac{A}{z+1}+\frac{B}{z-1}+\frac{C}{(z-1)^2}$ ……………….(1)

$Z^2-z+2=A(z-1)^2+B(z+1)(z-1)+C(z+1)$

Put z = -1,A=1,

Put z=1,C=1,

Putz=0,B=0.

Substitute the values of A,B and C in (1).

$\frac{X(z)}{z}=\frac{1}{z+1}+\frac{0}{z-1}+\frac{1}{(z-1)^2}$

$=\frac{1}{z+1}+\frac{1}{(z-1)^2}$

$X(Z)=\frac{z}{z+1}+\frac{z}{(z-1)^2}$

$Z\{x(n)\}=X(z)=\frac{z}{z+1}+\frac{z}{(z-1)^2}$

$X(n)=Z^{-1}\frac{z}{z+1}+Z^{-1}\frac{z}{(z-1)^2}$

$=(-1)^n+n$

Cauchy's residue theorem:

By Cauchy's residue theorem $\int_C X(z)z^{n-1}dz=2\pi$ i [sum of the residues of $X(z)z^{n-1}$ at the isolated singularities]

Problem-30:

Find $Z^{-1}\left[\left(\frac{10z}{(z-1)(z-2)}\right)\right]$

Solution

Let X(z)= $\left[\left(\frac{10z}{(z-1)(z-2)}\right)\right]$

$X(z)z^{n-1}= \left[\left(\frac{10z}{(z-1)(z-2)}\right)\right]z^{n-1}$

$=\left[\left(\frac{10z^n}{(z-1)(z-2)}\right)\right]$

Z=1 and z=2 are the simple poles.

$\underset{z=1}{\mathrm{Res}}\ X(z)z^{n-1}=\lim_{z\to 1}(z-1)\left[\left(\frac{10z^n}{(z-1)(z-2)}\right)\right]$

$=\lim_{z\to 1}\left[\left(\frac{10z^n}{(z-2)}\right)\right]$

=-10.

$\underset{z=2}{\mathrm{Res}}\ X(z)z^{n-1}=\lim_{z\to 1}(z-2)\left[\left(\frac{10z^n}{(z-1)(z-2)}\right)\right]$

$=\lim_{z\to 1}\left[\left(\frac{10z^n}{(z-1)}\right)\right]$

$=10\ 2^n$

x(n)= sum of the residues

$x(n)=10(2^n-1)$.

Problem-31:

Find $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

Solution:

Let X(z)= $\left[\frac{z^2}{(z-a)(z-b)}\right]$

$X(z)z^{n-1}=\left[\frac{z^2}{(z-a)(z-b)}\right]z^{n-1}$

z=a,z=b are simple poles.

$\underset{z=a}{\mathrm{Res}}\ X(z)z^{n-1}=\lim_{z\to a}\left[\frac{z^2}{(z-a)(z-b)}\right]z^{n-1}\ (z-a)$

$= \lim_{z \to a} \left[\frac{z^{n+1}}{(z-b)} \right]$

$= \left[\frac{a^{n+1}}{(a-b)} \right].$

$\operatorname{Res}_{z=b} X(z)z^{n-1} = \lim_{z \to b} \left[\frac{z^2}{(z-a)(z-b)} \right] z^{n-1} (z-b)$

$= \lim_{z \to b} \left[\frac{z^{n+1}}{(z-a)} \right]$

$= \left[\frac{b^{n+1}}{(b-a)} \right]$

x(n)=sum of residues.

$= \left[\frac{a^{n+1}}{(a-b)} \right] + \left[\frac{b^{n+1}}{(b-a)} \right].$

5.3 CONVOLUTION THEOREM:

The convolution theorem plays an important role in the solution of difference equation and in probability problem involving sums of two independent random variable.

Convolution of sequence:

1.The convolution of two sequence $\{x(n)\}$ & $\{y(n)\}$ is defined as

(i) $\{x(n) * y(n)\} = \sum_{K=-\infty}^{\infty} f(K)g(n-K)$ if the sequence are non-causal &

(ii) $\{x(n) * y(n)\} = \sum_{K=0}^{\infty} f(K)g(n-K)$ if the sequence are causal

2. The convolution of two sequence f(t) & g(t) is defined as f(t)*g(t) = $\sum_{K=0}^{\infty} f(KT)g(n-K)T$,where T is the samplieg period.

State and prove convolution theorem on Z-transform.

1.If $Z\{x(n)\} = X(z)$ & $Z\{y(n)\} = Y(z)$ then $Z\{x(n) * y(n)\} = X(z).Y(z)$

2.If $Z\{f(t)\} = F(z)$ & $Z\{g(t)\} = G(z)$ then $Z\{f(t) * g(t)\} = F(z).G(z)$

Proof:

1. $Z\{x(n) * y(n)\} = Z\left[\sum_{K=-\infty}^{\infty} x(K)y(n-K) \right]$

$= \sum_{k=-\infty}^{\infty} \left[\sum_{K=-\infty}^{\infty} x(K)y(n-K) \right] z^{-n}$

$= \sum_{k=-\infty}^{\infty} x(K) \sum_{K=-\infty}^{\infty} y(n-K)z^{-n}$

By changing the order of summation

$$= \sum_{k=-\infty}^{\infty} x(K)\left[\sum_{m=-\infty}^{\infty} y(m) z^{-(m+k)}\right]$$ by putting n-K = m

$$= \sum_{k=-\infty}^{\infty} x(K) z^{-k}\left[\sum_{m=-\infty}^{\infty} y(m) z^{-m}\right]$$

$$= \sum_{k=-\infty}^{\infty} x(K) z^{-k} \sum_{m=-\infty}^{\infty} y(m) z^{-m}$$

$$= X(z).Y(z)$$

$$\therefore Z\{x(n) * y(n)\} = X(z).Y(z)$$

2. $F(z)G(z) = \left[\sum_{K=o}^{\infty} f(KT) z^{-k}\right]\left[\sum_{r=o}^{\infty} g(rT) z^{-r}\right]$

$= \sum_{n=o}^{\infty} h(nT) z^{-n}$ say ……..(1)

when $h(nT) = f(0.T)g(nT) + f(1.T)g((n-1)T) + f(2.T)g((n-2)T) + ... + f(n.T)g(0.T)$

$= \sum_{K=o}^{n} f(KT) g\{(n-k)T\}$………..(2)

using (2) in (1) $F(z)G(z) = \sum_{n=0}^{\infty} z^{-n}\left[\sum_{K=o}^{n} f(KT) g\{(n-k)T\}\right]$

$$= \sum_{n=0}^{\infty} [f(t) * g(t)] z^{-n}$$

$$= Z[f(t) * g(t)]$$

$$F(z)G(z) = Z[f(t) * g(t)]$$

Problem-31:

Find $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

Solution:

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a} . \frac{z}{z-b}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n$$

$$= \sum_{m=0}^{n} a^m b^{n-m}$$

$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n \frac{1-\left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}} \qquad\qquad \because a + ar + ar^2 + \ldots + ar^n = \frac{a\left[1-r^n\right]}{1-r}, if r < 1$$

$$= \frac{b^{n+1} - a^{n+1}}{b-a}$$

$$\therefore Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{b^{n+1} - a^{n+1}}{b-a}$$

Problem-32:

Find $Z^{-1}\left[\frac{z^2}{(z+a)^2}\right]$

Solution:

$$Z^{-1}\left[\frac{z^2}{(z+a)^2}\right] = Z^{-1}\left[\frac{z}{z+a} . \frac{z}{z+a}\right]$$

$$= Z^{-1}\left[\frac{z}{z+a}\right] * z^{-1}\left[\frac{z}{z+b}\right]$$

$$= (-a)^n * (-a)^n$$

$$= \sum_{k=0}^{n} (-a)^K (-a)^{n-K}$$

$$= \sum_{k=0}^{n} (-a)^n$$

$$= (n+1)(-a)^n$$

$$\therefore Z^{-1}\left[\frac{z^2}{(z+a)^2}\right] = (n+1)(-a)^n .$$

5.4 Formation of difference equation

Difference equation: A difference equation is a relation between the differences of an unknown function at one or more general values of the argument. Thus $\Delta y(n+1) + y(n) = 2$ …(1) and $\Delta y(n+1) + \Delta^2 y(n-1) = 1$ …..(2) are difference equation.

Order of a difference equation:

The Order of a difference equation is the difference between the largest and the smallest arguments occurring in the difference equation divided by the unit of increment.

Solution of a difference equation:

The Solution of a difference equation is an expression for y(n) which satisfies the given difference equation.

The general Solution of a difference equation:

The general Solution of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.

The particular Solution of a difference equation:
The particular solution is that the solution which is obtained from the general solution by giving particular values to the constants.

Problem-33:

From $y_n = a2^n + b(-2)^n$,derive the difference equation by eliminating the constant.

Solution:

given $y_n = a2^n + b(-2)^n$ ……….(1)

$y_{n+1} = a2^{n+1} + b(-2)^{n+1} = 2a2^n - 2b(-2)^n$ ……….(2)

$y_{n+2} = a2^{n+2} + b(-2)^{n+2} = 4a2^n + 4b(-2)^n$ ……….(3)

Eliminating $a2^n$ & $b(-2)^n$ from (1) ,(2)&(3), we get

$$\Rightarrow \begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -2 \\ y_{n+2} & 4 & 4 \end{vmatrix} = 0$$

$y_n[8+8] - 1[4y_{n+1} + 2y_{n+2}] + 1[4y_{n+1} - 2y_{n+2}] = 0$

$16y_n - 4y_{n+1} - 2y_{n+2} + 4y_{n+1} - 2y_{n+2} = 0$

$16y_n - 4y_{n+2} = 0$

$y_{n+2} - 4y_n = 0$ which is the desired difference equation.

Problem-34:

Derive the difference equation from $y_n = (A + Bn)2^n$

Solution:

given $y_n = (A + Bn)2^n = A2^n + Bn2^n$ ……….(1)

$y_{n+1} = (A + B(n+1))2^{n+1} = 2A2^n + 2B(n+1)2^n$ ……….(2)

$y_{n+2} = (A + B(n+2))2^{n+2} = 4A2^n + 4B(n+2)2^n$ ……….(3)

Eliminating A & $B2^n$ from (1) ,(2)&(3), we get

$$\Rightarrow \begin{vmatrix} y_n & 1 & n \\ y_{n+1} & 2 & 2(n+1) \\ y_{n+2} & 4 & 4(n+2) \end{vmatrix} = 0$$

$y_n[8(n+2) - 8(n+1)] - 1[4(n+2)y_{n+1} - 2(n+1)y_{n+2}] + n[4y_{n+1} - 2y_{n+2}] = 0$

$y_n[8n + 16 - 8n - 8] - 1[(4n+8)y_{n+1} - (2n+2)y_{n+2}] + 4ny_{n+1} - 2ny_{n+2} = 0$

$8y_n - (4n+8)y_{n+1} + (2n+2)y_{n+2} + 4ny_{n+1} - 2ny_{n+2} = 0$

$y_{n+2}[-2n + (2n+2)] + y_{n+1}[-(4n+8) + 4n] + 8y_n = 0$

$2y_{n+2} - 8y_{n+1} + 8y_n = 0$

$y_{n+2} - 4y_{n+1} + 4y_n = 0$

5.5 Solution of difference equations using Z-Transform.

We know that Laplace Transform are very useful to solve linear differential equations.The Z-Transforms are useful to solve linear differential equations.

Formula:

1. $Z[y_n] = Y(z)$
2. $Z[y_{n+1}] = z\,Y(z) - z\,y(0)$
3. $Z[y_{n+2}] = z^2\,Y(z) - z^2\,y(0) - z\,Y(1)$
4. $Z[y_{n+3}] = z^3\,Y(z) - z^3\,y(0) - z^2\,Y(1) - z\,y(2)$
5. $Z[y_{n-1}] = z^{-1}Y(z)$.

Standard Formula:

$$1.Z[a^n] = \frac{z}{z-a}$$

$$2.Z[n] = \frac{z}{(z-1)^2}$$

$$3.Z[a^n n] = \frac{az}{(z-a)^2}$$

$$4.Z[n(n-1)] = \frac{2z}{(z-1)^3}$$

$$5.Z[a^n \cos n\theta] = \frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$$

$$6.Z[a^n \sin n\theta)] = \frac{az\sin\theta}{z^2-2az\cos\theta+a^2}$$

$$7.Z\left[\cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+1}$$

$$8.Z\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1}$$

Problem-35:

Solve $y_{n+1} - 2y_n = 0$ given $y_0 = 3$

Solution:

Given : $y_{n+1} - 2y_n = 0$

Taking Z- Transform on both sides of the difference equation ,we get $Z[y_{n+1}] - 2\,Z[y_n] = Z[0]$

$[zY(z) - zY(0)] - 2\,Y(z) = 0$

$[zY(z) - z(3)] - 2\,Y(z) = 0$ $\quad [\because y(0) = 3]$

$(z-2)Y(z) - 3z = 0$

$$Y(z) = \frac{3z}{z-2}$$

$$\Rightarrow Z[Y_n] = \frac{3z}{z-2}$$

$$Y_n = Z^{-1}\left[\frac{3z}{z-2}\right]$$

$$= 3Z^{-1}\left[\frac{z}{z-2}\right]$$

$$= 3\left(2^{n}\right) \qquad \because Z\left(a^{n}\right) = \frac{z}{z-a}$$

Problem-36:

Using Z-transform ,solve $u_{n+2} + 3u_{n+1} + 2u_n = 0$ given $u_0 = 1$, $u_1 = 2$

Solution:

Given $u_{n+2} + 3u_{n+1} + 2u_n = 0$

$Z[u_{n+2}] + 3Z[u_{n+1}] + 2Z[u_n] = 0$

$[z^2u(z) - z^2u(0) - z\,u(1)] + 3[zu(z) - zu(0)] + 2u(z) = 0$

$(Z^2 + 3z + 2)u(z) - z^2 - 2z - 3z = 0 \quad \because u(0) = 1, u(1) = 2$

$(Z^2 + 3z + 2)u(z) - z^2 - 5z = 0$

$$U(z) = \frac{z^2 + 5z}{z^2 + 3z + 2}$$

$$U(z) = \frac{z(z+5)}{(z+1)(z+2)}$$

$$\frac{U(z)}{z} = \frac{(z+5)}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2} \text{ ....(1)}$$

$(z+5) = A(z+2) + B(z+1)$

Put z = -1, we get A = 4

Put z = -2, we get B = -3

$$\therefore (1) \Rightarrow \frac{U(z)}{z} = \frac{4}{z+1} + \frac{-3}{z+2}$$

$$U(z) = 4\left[\frac{z}{z+1}\right] - 3\left[\frac{z}{z+2}\right]$$

$$ZU(n) = 4\left[\frac{z}{z+1}\right] - 3\left[\frac{z}{z+2}\right]$$

$$U(n) = 4Z^{-1}\left[\frac{z}{z+1}\right] - 3Z^{-1}\left[\frac{z}{z+2}\right]$$

$$U(n) = 4(-1)^n - 3(-2)^n$$

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SUBJECT NAME : **Transforms and Partial Differential Equations**

SUBJECT CODE : **MA2211**

MATERIAL NAME : **FormulaMaterial**

MATERIAL CODE : **JM08AM3005**

Unit – I (Fourier Series)

1) Dirichlet's Conditions:

Any function $f(x)$ can be expanded as a Fourier series $\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ where a_0, a_n, b_n are constants provided the following conditions are true.

- $f(x)$ is periodic, single – valued and finite.
- $f(x)$ has a finite number of discontinuities in any one period.
- $f(x)$ has at the most a finite number of maxima and minima.

2) The Fourier Series in the interval (0,2π):

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

Where $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx$, $a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx dx$, $b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx dx$

3) The Fourier Series in the interval (-π,π):

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

Where $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$, $a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx dx$, $b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx dx$

In this interval, we have to verify the function is either odd function or even function. If it is even function then find only a_0 and a_n ($b_n=0$). If it is odd function then find only b_n ($a_0=a_n=0$).

If the function is neither odd nor even then you should find a_0, a_n **and** b_n by using the following formulas $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$.

4) The half range Fourier Series in the interval (0,π):
   - The half range Cosine Series in the interval (0,π):

     $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

     Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$
   - The half range Sine Series in the interval (0,π):

     $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

     Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$

5) The Parseval's Identity in the interval (0,2π):

$$\frac{1}{\pi}\int_0^{2\pi}[f(x)]^2 dx = \frac{a_0^2}{4} + \sum_{n=1}^{\infty}[a_n^2 + b_n^2]$$

6) The Parseval's Identity in the interval (-π,π):

$$\frac{2}{\pi}\int_0^{\pi}[f(x)]^2 dx = \frac{a_0^2}{4} + \sum_{n=1}^{\infty}[a_n^2 + b_n^2]$$

7) The Parseval's Identity for half range cosine series in the interval (0,π):

$$\frac{2}{\pi}\int_0^{\pi}[f(x)]^2 dx = \frac{a_0^2}{4} + \sum_{n=1}^{\infty} a_n^2$$

8) The Parseval's Identity for half range sine series in the interval (0,π):

$$\frac{2}{\pi}\int_0^{\pi}[f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2$$

Change of interval:

9) The Fourier Series in the interval (0,2ℓ):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

Where $a_0 = \frac{1}{\ell}\int_0^{2\ell} f(x)dx$, $a_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\cos\frac{n\pi x}{\ell}dx$, $b_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{\ell}dx$

10) The Fourier Series in the interval (-ℓ, ℓ):

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$$

Where $a_0=\frac{2}{\ell}\int_0^{\ell}f(x)dx$, $a_n=\frac{2}{\ell}\int_0^{\ell}f(x)\cos\frac{n\pi x}{\ell}dx$, $b_n=\frac{2}{\ell}\int_0^{\ell}f(x)\sin\frac{n\pi x}{\ell}dx$

In this interval, you have to verify the function is either odd function or even function. If it is even function then find only a_0 **and** a_n ($b_n=0$). If it is odd function then find only b_n ($a_0=a_n=0$).

11) The half range Fourier Series in the interval (0, ℓ):

- The half range Cosine Series in the interval (0, ℓ):

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}$$

Where $a_0=\frac{2}{\ell}\int_0^{\ell}f(x)dx$, $a_n=\frac{2}{\ell}\int_0^{\ell}f(x)\cos\frac{n\pi x}{\ell}dx$

- The half range Sine Series in the interval (0, ℓ):

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$$

Where $b_n=\frac{2}{\ell}\int_0^{\ell}f(x)\sin\frac{n\pi x}{\ell}dx$

12) The Parseval's Identity in the interval (0,2ℓ):

$$\frac{1}{\ell}\int_0^{2\ell}[f(x)]^2dx=\frac{a_0^2}{4}+\sum_{n=1}^{\infty}[a_n^2+b_n^2]$$

13) The Parseval's Identity in the interval (-ℓ,ℓ):

$$\frac{2}{\ell}\int_0^{\ell}[f(x)]^2dx=\frac{a_0^2}{4}+\sum_{n=1}^{\infty}[a_n^2+b_n^2]$$

14) The Parseval's Identity for half range cosine series in the interval (0,ℓ):

$$\frac{2}{\ell}\int_0^{\ell}[f(x)]^2dx=\frac{a_0^2}{4}+\sum_{n=1}^{\infty}a_n^2$$

15) The Parseval's Identity for half range sine series in the interval (0,ℓ):

$$\frac{2}{\ell}\int_0^{\ell}[f(x)]^2dx=\sum_{n=1}^{\infty}b_n^2$$

16) Harmonic Analysis:

The method of calculation of Fourier constants by means of numerical calculation is called as Harmonic analysis.

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

where

$$a_0=\frac{2}{n}\sum y \;,\quad a_1=\frac{2}{n}\sum y\cos x \;,\quad a_2=\frac{2}{n}\sum y\cos 2x \;,\quad a_3=\frac{2}{n}\sum y\cos 3x \;,\ldots$$

$$b_1=\frac{2}{n}\sum y\sin x \;,\quad b_2=\frac{2}{n}\sum y\sin 2x \;,\quad b_3=\frac{2}{n}\sum y\sin 3x \;,\ldots$$

When the values of x is given as numbers the θ is calculated by $\theta=\frac{2\pi x}{T}$.

Where T is period, n is the number of values given. If the first and last y values are same we can omit one of them.

Complex form of Fourier Series:

17) The Complex form of Fourier Series in the interval (0,2π):

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{inx} \qquad \text{where } c_n=\frac{1}{2\pi}\int_0^{2\pi}f(x)e^{-inx}dx$$

18) The Complex form of Fourier Series in the interval (-π,π):

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{inx} \qquad \text{where } c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx$$

19) The Complex form of Fourier Series in the interval (0,2ℓ):

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{\frac{in\pi x}{\ell}} \qquad \text{where } c_n=\frac{1}{2\ell}\int_0^{2\ell}f(x)e^{\frac{-in\pi x}{\ell}}dx$$

20) The Complex form of Fourier Series in the interval (-ℓ,ℓ):

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{\frac{in\pi x}{\ell}} \qquad \text{where } c_n=\frac{1}{2\ell}\int_{-\ell}^{\ell}f(x)e^{\frac{-in\pi x}{\ell}}dx$$

Unit – II (Fourier Transforms)

1) Fourier Integral theorem

The Fourier integral theorem of $f(x)$ in the interval $(-\ell,\ell)$ is

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(x)\cos\lambda(t-\lambda)dxd\lambda$$

2) Convolution Theorem

If $F[s]$ and $G[s]$ are the Fourier transform of the functions $f(x)$ and $g(x)$ respectively, then $F[f(x)*g(x)] = F[s].G[s]$

3) The Fourier Transform of a function $f(x)$ is given by $F[f(x)]$ is denoted by $F[s]$.

4) Fourier Transform $F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

5) Inverse Fourier Transform $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[s] e^{-isx} ds$

6) The Fourier transforms and Inverse Fourier transforms are called Fourier transforms pairs.

7) Fourier Sine Transform $F_s[s] = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$

8) Fourier Cosine Transform $F_c[s] = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$

9) If $f(x) = e^{-ax}$ then the Fourier Cosine and Sine transforms as follows

   a) $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2}$

   b) $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2}$

10) Property

   a) $F_s[xf(x)] = -\frac{d}{ds} F_c[f(x)]$

   b) $F_c[xf(x)] = \frac{d}{ds} F_s[f(x)]$

11) Parseval's Identity

   a) $\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$

   b) $\int_0^{\infty} F_c(s) G_c(s) ds = \int_0^{\infty} f(x) g(x) dx \quad \text{(Or)} \quad \int_0^{\infty} |F_c(s)|^2 ds = \int_0^{\infty} |f(x)|^2 dx$

12) Condition for Self reciprocal $F[f(x)] = f(s)$

Unit – III (Partial Differential Equation)

1) Lagrange's Linear equation

   The equation of the form $Pp + Qq = R$

   then the subsidiary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

2) Homogeneous Linear Partial Differential Equation of higher order with constant coefficients:

   The equation of the form $a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} = f(x, y)$

   The above equation can be written as

   $$\left(aD^2 + bDD' + cD'^2\right)z = f(x, y) \quad \ldots\ldots\ldots\ldots (1)$$

   where $D^2 = \frac{\partial^2}{\partial x^2}$, $D = \frac{\partial}{\partial x}$ and $D'^2 = \frac{\partial^2}{\partial y^2}$, $D' = \frac{\partial}{\partial y}$

   The solution of above equation is z = C.F + P.I

   - **Complementary Function (C.F) :**

     To find C.F consider the auxiliary equation by replacing D by m and D' by 1.The equation (1) implies that $am^2 + bm + c = 0$, solving this equation we get two values of m. The following table gives C.F of the above equation.

Sl.No.	Nature of m	Complementary Function
1	$m_1 \neq m_2$	C.F $= f_1(y + m_1x) + f_2(y + m_2x)$
2	$m_1 = m_2$	C.F $= f_1(y + mx) + xf_2(y + mx)$
3	$m_1 \neq m_2 \neq m_3$	C.F $= f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x)$
4	$m_1 = m_2 = m_3$	C.F $= f_1(y + mx) + xf_2(y + mx) + x^2 f_3(y + mx)$
5	$m_1 = m_2,\quad m_3$ is different	C.F $= f_1(y + mx) + xf_2(y + mx) + f_3(y + m_3x)$

   - **Particular Integral (P.I) :**

     To find P.I consider $\phi(D, D') = aD^2 + bDD' + cD'^2$.

     Type: 1If $f(x, y) = 0$, then $\text{P.I} = 0$.

     Type: 2 If $f(x, y) = e^{ax+by}$

     $$P.I = \frac{1}{\phi(D, D')} e^{ax+by}$$

Replace D by a and D' by b. If $\phi(D,D') \neq 0$, then it is P.I. If $\phi(D,D')=0$, then diff. denominator w.r.t D and multiply x in numerator. Again replace D by a and D' by b. If again denominator equal to zero then continue the same procedure.

Type: 3 If $f(x,y)=\sin(ax+by)$ (or) $\cos(ax+by)$

$$P.I=\frac{1}{\phi(D,D')}\sin(ax+by) \text{ (or) } \cos(ax+by)$$

Here replace D^2 by $-a^2$, D'^2 by $-b^2$ and DD' by $-ab$. Do not replace for D and D'. If the denominator equal to zero, then apply the same producer as in Type: 2.

Type: 4 If $f(x,y)=x^m y^n$

$$P.I=\frac{1}{\phi(D,D')}x^m y^n$$
$$=\frac{1}{1+g(D,D')}x^m y^n$$
$$=\left(1+g(D,D')\right)^{-1}x^m y^n$$

Here we can use Binomial formula as follows:

i) $(1+x)^{-1}=1-x+x^2-x^3+\dots$

ii) $(1-x)^{-1}=1+x+x^2+x^3+\dots$

iii) $(1+x)^{-2}=1-2x+3x^2-4x^3+\dots$

iv) $(1-x)^{-2}=1+2x+3x^2+4x^3+\dots$

v) $(1+x)^{-3}=1-3x+6x^2-10x^3+\dots$

vi) $(1-x)^{-3}=1+3x+6x^2+10x^3+\dots$

Type: 5 If $f(x,y)=e^{ax+by}\mathrm{V}$, where

$\mathrm{V}=\sin(ax+by)$ (or) $\cos(ax+by)$ (or) $x^m y^n$

$$P.I=\frac{1}{\phi(D,D')}e^{ax+by}\mathrm{V}$$

First operate e^{ax+by} by replacing D by $D+a$ and D' by $D'+a$.

$P.I=e^{ax+by}\dfrac{1}{\phi(D+a,D'+b)}\mathrm{V}$, Now this will either Type: 3 or Type: 4.

Type: 6 If $f(x,y) = y\sin ax$ (or) $y\cos ax$

$$P.I = \frac{1}{\phi(D,D')} y\sin ax$$

$$= \frac{1}{(D-m_1D')(D-m_2D')} y\sin ax \qquad y \to c - m_2 x$$

$$= \frac{1}{(D-m_1D')} \int (c-m_2x)\sin ax\, dx \text{ (Apply Bernouili's method)}$$

3) Solution of Partial Differential Equations:

Standard Type: 1 Equation of the form $f(p,q) = 0$

Assume that $z = ax + by + c$ be the solution the above equation.put $p = a$ and $q = b$ in equation (1), we get $f(a,b) = 0$. Now, solve this, we get $b = \phi(a)$.
$z = ax + \phi(a)y + c$ which is called Complete solution.

Standard Type: 2 Equation of the form $z = px + qy + f(p,q)$ (Clairaut's form)

The Complete solution is $z = ax + by + f(a,b)$. To find Singular integral diff. partially w.r.t a & b, equate to zero and eliminate a and b.

Standard Type: 3 Equation of the form $f_1(x,p) = f_2(y,q)$

The solution is $z = \int p\,dx + \int q\,dy$.

Standard Type: 4 Equation of the form $f(z,p,q) = 0$

In this type put $u = x + ay$, then $p = \frac{dz}{du}, q = a\frac{dz}{du}$

Unit – IV (Application of Partial Differential Equation)

1) The One dimensional Wave equation:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The three solutions of the above equation are

i) $y(x,t) = (Ae^{px} + Be^{-px})(Ce^{pat} + De^{-pat})$

ii) $y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$

iii) $y(x,t) = (Ax + B)(Ct + D)$

But the correct solution is ii),

$y(x,t)=\left(A\cos px+B\sin px\right)\left(C\cos pat+D\sin pat\right)$.

2) The One dimensional Heat flow equation:

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$$

$\alpha^2=\frac{k}{\rho c}$ where $k=$ **Thermal Conductivity**, $\rho=$ **Density**, $c=$ **Specific Heat**

The three solutions of the above equation are

i) $u(x,t)=\left(Ae^{px}+Be^{-px}\right)Ce^{\alpha^2p^2t}$

ii) $u(x,t)=\left(A\cos px+B\sin px\right)Ce^{-\alpha^2p^2t}$

iii) $u(x,t)=\left(Ax+B\right)C$

But the correct solution is ii), $u(x,t)=\left(A\cos px+B\sin px\right)Ce^{-\alpha^2p^2t}$

3) The Two dimensional Heat flow equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

The three solutions of the above equation are

i) $u(x,y)=\left(Ae^{px}+Be^{-px}\right)\left(C\cos py+D\sin py\right)$

(Applicable when given value is parallel to y-axies)

ii) $u(x,y)=\left(A\cos px+B\sin px\right)\left(Ce^{py}+De^{-py}\right)$

(Applicable when given value is parallel to x-axies)

iii) $u(x,y)=\left(Ax+B\right)\left(Cy+D\right)$ (Not applicable)

Unit – V (Z - Transform)

1) Definition of Z-transform:

Let $\left\{f(n)\right\}$ be the sequence defined for all the positive integers n such that

$$Z\left[f(n)\right]=\sum_{n=0}^{\infty}f(n)z^{-n}$$

2)

Sl.No	$Z[f(n)]$	$F[z]$
1.	$Z[1]$	$\frac{z}{z-1}$
2.	$Z[(-1)^n]$	$\frac{z}{z+1}$
3.	$Z[a^n]$	$\frac{z}{z-a}$
4.	$Z[n]$	$\frac{z}{(z-1)^2}$
5.	$Z[n+1]$	$\frac{z^2}{(z-1)^2}$
6.	$Z\left[\frac{1}{n}\right]$	$\log\left(\frac{z}{z-1}\right)$
7.	$Z\left[\sin\frac{n\pi}{2}\right]$	$\frac{z}{z^2+1}$
8.	$Z\left[\cos\frac{n\pi}{2}\right]$	$\frac{z^2}{z^2+1}$

3) Statement of Initial value theorem:

If $Z[f(n)] = F[z]$, then $\underset{z\to\infty}{Lt} F[z] = \underset{n\to 0}{Lt} f(n)$

4) Statement of Final value theorem:

If $Z[f(n)] = F[z]$, then $\underset{n\to\infty}{Lt} f(n) = \underset{z\to 1}{Lt}(z-1)F(z)$

5) $Z[a^n f(n)] = \left(Z[f(n)]\right)_{z\to\frac{z}{a}}$

6) $Z[nf(n)] = -z\frac{d}{dz}Z[f(n)]$

7) Inverse Z-transform

Sl.No	$Z^{-1}[F(z)]$	$f(n)$
1.	$Z^{-1}\left(\frac{z}{z-1}\right)$	1
2.	$Z^{-1}\left(\frac{z}{z+1}\right)$	$(-1)^n$
3.	$Z^{-1}\left(\frac{z}{z-a}\right)$	a^n

4.	$Z^{-1}\left(\frac{z}{z+a}\right)$	$(-a)^n$
5.	$Z^{-1}\left(\frac{z}{(z-1)^2}\right)$	n
6.	$Z^{-1}\left(\frac{z}{(z-a)^2}\right)$	na^{n-1}
7.	$Z^{-1}\left(\frac{z}{(z+a)^2}\right)$	$n(-a)^{n-1}$
8.	$Z^{-1}\left(\frac{z^2}{z^2+1}\right)$	$\cos\frac{n\pi}{2}$
9.	$Z^{-1}\left(\frac{z^2}{z^2+a^2}\right)$	$a^n\cos\frac{n\pi}{2}$
10.	$Z^{-1}\left(\frac{z}{z^2+1}\right)$	$\sin\frac{n\pi}{2}$
11.	$Z^{-1}\left(\frac{z}{z^2+a^2}\right)$	$a^{n-1}\sin\frac{n\pi}{2}$

8) Inverse form of Convolution Theorem

$$Z^{-1}[F(z).G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$$

and by the defn. of Convolution of two functions $f(n)*g(n) = \sum_{r=0}^{n} f(r)g(n-r)$

9) a) $Z[y(n)] = F(z)$

b) $Z[y(n+1)] = zF(z) - zy(0)$

c) $Z[y(n+2)] = z^2F(z) - z^2y(0) - zy(1)$

d) $Z[y(n+3)] = z^3F(z) - z^3y(0) - z^2y(1) - zy(2)$

----All the Best----